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ON THE ψ-PRODUCT OF D. H. LEHMER-II

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If F denotes the set of arithemetical functions, we consider the commutative ring $(F, +, \psi)$ where ψ is the modified Lehmer ψ-product. If L is a non-empty subset of positive integers and X is the characteristic function of L, in this paper, among other things, we investigate the problem of determining necessary and sufficient conditions for the function X to be the unity of $(F, +, \psi)$ for some ψ. We show that this is possible if and only if $1 \in L$.

§1. Let F denote the set of all arithmetrical functions and T be a non-empty subset of $Z^+ \times Z^+$, where Z^+ is the set of positive integers. Let $\psi : T \to Z^+$ be a mapping satisfying the following :

(I) For each $n \in Z^+$, $\psi(x, y) = n$ has a finite number of solutions.

(II) If $(x, y) \in T$, then $(y, x) \in T$ and $\psi(x, y) = \psi(y, x)$.

(III) "$(y, z) \in T$ and $(x, \psi(y, z)) \in T$" if and only if

"$(x, y) \in T$ and $(\psi(x, y), z) \in T$"; $\psi(x, \psi(y, z))$

$= \psi(\psi(x, y), z)$, whenever one of these conditions holds.

(IV) $\psi(1,1) = 1$ and for each $k \in Z^+$, $\psi(x, k) = k$ has a solution.

(V) For each $k \in Z^+$, $k = \max\{x \in Z^+ : \psi(x, y) = k \text{ for some } y \in Z^+\}$ or equivalently $\psi(x, y) \geqslant \max\{x, y\}$, for all $(x, y) \in T$.

Let

$$S_k = \{x \in Z^+ : \psi(x, k) = k\} \qquad \ldots(1.1)$$

and

$$i_k = \min S_k, \qquad \ldots(1.2)$$

for each $k \in Z^+$. Also, let

$$S = \{i_k : k = 1, 2, 3, \ldots\}. \qquad \ldots(1.3)$$

(VI) For each $k, n \in Z^+$, the equation $\psi(x, k) = n$ has a unique solution in S if $n = k$ and no solutions in S if $n \neq k$.

If we define the binary operation ψ on F by

$$(f \psi g)(n) = \sum_{\psi(x,y)=n} f(x)\, g(y) \qquad \ldots(1.4)$$

for every $n \in Z^+$ and $f, g \in F$, then we[1] proved, that $(F, +, \psi)$ is a commutative ring with unity X, where X is the characteristic function of the set S given in (1.3).

Suppose L is a non-empty subset of positive integers and let X be the chnracteristic function of L. Then the question is: can we find a non-empty subset $T \subseteq Z^+ \times Z^+$ and a function $\psi : T \to Z^+$ satisfying (I) through (V) such that X is the unity of the ring $(F, +, \psi)$? In section 2 (see Remark 2.4) we show that the answer to this question is positive if and only if $1 \in L$.

In section 3, we show that the unity of $(F, +, \psi)$ must be an integer-valued function and if it is non-negative, then it must be the characteristicfu nction of the set S given in (1.3). We also give an example of a ring $(F, +, \psi)$ whose unity assumes negative values also (see example 3.1).

§2. We need the following :

Lemma 2.1 (*cf.* Sitaramaiah[1], Lemma 2.1)—Suppose ψ satisfies the condition (I) through (IV) of section 1. An arithmetical function g is an identity with respect to ψ, that is,

$$f \psi g = f, \text{ for all } f \in F,$$

if and only if, for any fixed $k, n \in Z^+$,

$$\sum_{\psi(i,k)=n} g(i) = \begin{cases} 1, & \text{if } n = k \\ 0, & \text{if } n \neq k. \end{cases}$$

Now we prove :

Theorem 2.1—Let L be a non-empty subset of positive integers and X be the characteristic function of L. Suppose we can find $\phi \neq T \subseteq Z^+ \times Z^+$, a function $\psi : T \to Z^+$ which satisfies the conditions (I) through (V) of § 1, and such that X is the unity of the ring $(F, +, \psi)$. Then there exists a mapping $\alpha : Z^+ \to L$ such that

(a) $L = \{\alpha(k) : k = 1, 2, 3, \ldots\}$

(b) $\alpha(\alpha(k)) = \alpha(k)$, for all $k \in Z^+$

(c) $\alpha(k) \leqslant k$, for all $k \in Z^+$.

Moreover, we have

(d) $(x, y) \in T \Rightarrow \alpha(x) = \alpha(y) = \alpha(\psi(x, y)$

(e) $i_k = \alpha(k)$, where i_k is given by (1.2).

PROOF : Since X is the unity of $(F, +, \psi)$, using Lemma 2.1, we see that for k, $n \in Z^+$ the equation $\psi(x, k) = n$ has a unique solution in L if $n = k$ and no solutions in L if $n \neq k$. So, if $k \in Z^+$, then there exists a unique element $\alpha(k)$ of L such that $\psi(\alpha(k), k) = k$. Hence $\{\alpha(k) : k = 1, 2, 3, \ldots\} \subseteq L$. Let $n \in L$. We have $\psi(n, \alpha(n)) = n$. Put $k = \alpha(n)$ so that $\psi(n, k) = n$. Since the equation $\psi(x, k) = n$ has no solution in L if $k \neq n$, we must have $k = n$. That is, $n = \alpha(n)$ so that $L \subseteq \{\alpha(k) : k = 1, 2, 3, \ldots\}$. Hence (a) follows.

Since $(\alpha(k), k) \in T$ for every $k \in Z^+$, so is $(\alpha(\alpha(k)), \alpha(k))$ for any $k \in Z^+$. Now using the condition (III) of section 1, we get

$$
\begin{aligned}
k = \psi(k, \alpha(k)) &= \psi(k, \psi(\alpha(k), \alpha(\alpha(k)))) \\
&= \psi(\psi(k, \alpha(k)), \alpha(\alpha(k))) \\
&= \psi(k, \alpha(\alpha(k))).
\end{aligned}
$$

Since $\alpha(\alpha(k)) \in L$ and $\psi(x, k) = k$ has a unique solution in L, we must have $\alpha(\alpha(k)) = \alpha(k)$. Hence (b) follows.

Since $\psi(\alpha(k), k) = k$, (c) follows from the condition (V) of § 1. To prove (d), let $(x, y) \in T$ and $k = \psi(x, y)$. Since ψ satisfies the condition (II) and (III) of § 1, we must have

$$
\begin{aligned}
k = \psi(x, y) &= \psi(x, \alpha(x)), y) \\
&= \psi(\alpha(x), \psi(x, y)) \\
&= \psi(\alpha(x), k).
\end{aligned}
$$

Hence $\alpha(x) = \alpha(k)$. Similarly $\alpha(y) = \alpha(k)$.

Therefore, $\alpha(x) = \alpha(y) = \alpha(\psi(x, y))$.

Finally, by (d) we have

$$
S_k = \{x : \psi(x, k) = k\} = \{x : \psi(x, k) = k \\ \alpha(x) = \alpha(k)\}.
$$

Clearly $\alpha(k) \in S_k$ and if $x \in S_k$, then by (c), $\alpha(k) = \alpha(x) \leqslant x$.

Thus $\alpha(k) = \min S_k$. Hence (e) follows.

Remark 2.1 : From (c) and (a) of Theorem 2.1 we have $\alpha(1) = 1$ and $1 \in L$.

Theorem 2.2—Let $L = \{\alpha(k): k = 1, 2, 3, \ldots\}$ where (i) $\alpha(k) \leqslant k$ and (ii) $\alpha(\alpha(k)) = \alpha(k)$, for all $k \in Z^+$. Let $\phi \neq T \subseteq \{(x, y) \in Z^+ \times Z^+: \alpha(x) = \alpha(y)\}$ and $\psi: T \to Z^+$ satisfy the conditions (I) through (V) of § 1. If $(k, \alpha(k)) \in T$ for every $k \in Z^+$ and $\psi(k, \alpha(k)) = k$, then the corresponding ring $(F, +, \psi)$ has the unity X, where X is the characteristic function of L.

PROOF : By hypothesis, $\psi(\alpha(k), k) = k$ and if $\psi(\alpha(r), k) = k$, then $\alpha(\alpha(r)) = \alpha(k)$ so that $\alpha(r) = \alpha(k)$. Hence $\psi(x, k) = k$ has a unique solution in L. If $k \neq n$ and $\psi(\alpha(r), k) = n$, then we must have $\alpha(\alpha(r)) = \alpha(k)$ or $\alpha(r) = \alpha(k)$. But then $\psi(\alpha(r), k) = \psi(\alpha(k), k) = k \neq n$. Hence $\psi(x, k) = n$ has no solutions in L. Therefore, X is the unity of $(F, +, \psi)$.

Corallary 2.1—Let $L = \{\alpha(k) : k = 1, 2, 3, \ldots\}$, where $\alpha : Z^+ \to Z^+$ is a mapping satisfying

$$\alpha(\alpha(k)) = \alpha(k) \quad \ldots(2.1)$$

and

$$\bar{\alpha}(k) \leqslant k \quad \ldots(2.2)$$

for each $k \in Z^+$. Let $T = \{(x, y) : Z^+ \times Z^+ : \alpha(x) = \alpha(y)\}$ and $\psi : T \to Z^+$ be defined by $\psi(x, y) = \max\{x, y\}$. If the binary operation ψ is as given in (1.4), then the triple $(F, +, \psi)$ is a commutative ring with unity X, where X is the characteristic function of L.

PROOF : It is easy to verify that T and ψ satisfy the hypothesis of Theorem 2.2. Hence Corollary 2.1 follows :

Remark 2.2 : Let $L = \{\alpha(k) : k = 1, 2, \ldots\}$, where $\alpha(k) \leqslant k$ and $\alpha(\alpha(k)) = \alpha(k)$, for all $k \in Z^+$. Let X be the characteristic function of L and $T = \{(x, y) \in Z^+ \times Z^+ : \alpha(x) = \alpha(y)\}$. Let $\psi : T \to Z^+$ be defined by $\psi(x, y) = [x, y]$ = The least common multiple of x and y. If ψ satisfies the conditions (I) through (V) of § 1 and the triple $(F, +, \psi)$ has X as the unity, then by (e) of Theorem 2.1 we must have $\alpha(k) = \min\{x \in Z^+ : \psi(x, k) = k\} = \min\{x \in Z^+ : [x, k] = k, \alpha(x) = \alpha(k)\}$, which implies that $\alpha(k) \mid k$. Also, by (d) of Theorem 2.1, $(x, y) \in T \Rightarrow \alpha(x) = \alpha(y) = \alpha(\psi(x, y)) = \alpha([x, y])$. That is, $\alpha(x) = \alpha(y) \Rightarrow \alpha(x) = \alpha([x, y])$. Conversely, if $T = \{(x,y) \in Z^+ \times Z^+ : \alpha(x) = \alpha(y)\}$ where (i) $\alpha(k) \mid k$ (ii) $\alpha(\alpha(k)) = \alpha(k)$, for all $k \in Z^+$ and (iii) $\alpha(x) = \alpha(y) \Rightarrow \alpha(x) = \alpha([x, y])$, then $\psi : T \to Z^+$ defined by $\psi(x, y) = [x, y]$, satisfies the conditions (I) through (V) of section 1. Hence using Theorem 2.2, we find that $(F, +, \psi)$ is a commutative ring with unity X, where X is the characteristic function of the set $\{\alpha(k); k = 1, 2, 3, \ldots\}$.

Remark 2.3 : Let $\gamma(1) = 1$ and for $k > 1$, $\gamma(k)$ = the product of distinct prime factors of k. Then clearly $\gamma(k) \mid k$ and $\gamma(\gamma(k)) = \gamma(k)$ for all $k \in Z^+$. If $\Gamma = \{(x, y) \in Z^+ \times Z^+ : \gamma(x) = \gamma(y)\}$, then by taking $\alpha(k) = \gamma(k)$ for all k, from Corollary 2.1 and Remark 2.2, we see that $(F, +, \psi_1)$ and $(F, +, \psi_2)$ are commutative rings with unity = The characteristic function of $\{\gamma(k) = k = 1, 2, 3 \ldots\}$ = The characteristic function of the square free integers = $|\mu|$, μ being the Möbius function, where $\psi_1(x,y) = \max\{x, y\}$ and $\psi_2(x, y) = [x, y]$, for all $(x, y) \in T$. These two examples have already been discussed (see Examples 3.10 and 3.11 of Sitramaiah[1]).

Remark 2.4 : Let L be a nonempty set of positive integers. If $1 \notin L$, by Remark 2.1, it is not possible to find a function ψ which satisfies the conditions (I) through (V) of section 1 such that the characteristic function of L is the unity of the ring $(F, +, \psi)$. If $1 \in L$ and $L = \{\alpha(k) : k = 1, 2, \ldots,\}$ where $\alpha : Z^+ \to Z^+$ is a mapping satisfying $\alpha(\alpha(k)) = \alpha(k)$ and $\alpha(k) \leqslant k$ for all $k \in Z^+$ then Corollary 2.1 shows that it is possible to find a function ψ satisfying the conditions (I) through (V) such that the commutative ring $(F, +, \psi)$ has the unity X, where X is the characteristic function of L. If $1 \in L$, we can always find $\alpha : Z^+ \to L$ satisfying $\alpha(\alpha(k)) = \alpha(k)$ and $\alpha(k) \leqslant k$ for all $k \in Z^+$. For, first we assume that L is finite, say $L = \{x_1, x_2, \ldots x_n\}$ with $1 = x_1 < x_2 < \ldots < x_n$. Let us define $\alpha : Z^+ \to L$ by $\alpha(x) = x_j$, if $x = x_j$, and $1 \leqslant j \leqslant n$, $\alpha(x) = x_j$, if $x_j < x < x_{j+1}$, $1, \leqslant j \leqslant n-1$ and $\alpha(x) = x_n$, if $x > x_n$, If $1 \in L$ and L is an infinite set of positive integers we can always write $L = \{x_1, x_2 \ldots\}$, where $x_1 = 1$ and $x_j < x_{j+1}$, for $j = 1, 2, 3, \ldots$, we may define $\alpha : Z^+ \to L$ by $\alpha(x) = x_j$, if $x = x_j$ and $\alpha(x) = x_r$, if $x_r < x < x_{r+1}$, for some positive integer r. In both the cases, it is clear that $\alpha(\alpha(k)) = \alpha(k)$ and $\alpha(k) \leqslant k$ for all $k \in Z^+$. Thus if $1 \in L$, by corollary 2.1, it is always possible to find *a* ψ such that $(F, +, \psi)$ is a commutative ring with unity X, where X is the characteristic function of L.

Remark 2.5 : In all the examples discussed in section 3 of Sitaramaiah[1], the corresponding rings $(F, +, \psi)$ have a unity which is a multiplicative function ($f \in F$ is said to be multiplicative if $f \not\equiv 0$ and $f(mn) = f(m) f(n)$ whenever m and n relatively prime integers). It is easy to give examples of commutative rings $(F, +, \psi)$ whose unity need not be multiplicatative. For example, if $L = \{1, 2, 3\}$, and the ring $(F, +, \psi)$ is such that its unity is the characteristic function X of L, then clearly X is not multtplicative since $X(6) = 0 \neq X(2) X(3) = 1$. Of course, if X is the characteristic function of a non-empty set of positive integers L, then X is multiplicative if and only if L is a multiplicative set (that is, if m and n are relatively prime, then $mn \in L$ if and only if $m \in L$ and $n \in L$).

Remark 2.6 : We observe that if $L = \{\alpha(k) : k = 1, 2, \ldots\}$, where (i) $\alpha(k) \leqslant k$ (ii) $\alpha(\alpha(k)) = \alpha(k)$, for all $k \in Z^+$ and α is multiplicative, then L is a multiplicative set (so the characteristic function of L is multiplicative). To see this, let m and n be relatively prime. If $m, n \in L$, then $\alpha(m) = m$ and $\alpha(n) = n$. Since α is multiplicative, $\alpha(mn) = \alpha(m)\alpha(n) = mn$, so that $mn \in L$. Conversely, if $mn \in L$, then $mn = \alpha(mn) = \alpha(m)\alpha(n)$; since $\alpha(m) \leqslant m$ and $\alpha(n) \leqslant n$, We must have $\alpha(m) = m$ and $\alpha(n) = n$. So: $m, n \in L$.

Remark 2.7 : In all the examples discussed in section 3 of Sitaramaih[1] in the corresponding rings $(F, +, \psi)$, $i_k \mid k$ for all $k \in Z^+$, where i_k is given by (1.2). It is easy to contruct an example of a ring $(F, +, \psi)$ in which $i_k \nmid k$ for some $k \in Z^+$. For example, if $L = \{1,2,4\}$, and $\alpha(1) = 1$, $\alpha(2) = 2$, $\alpha(3) = 2$, $\alpha(4) = 4$ and $\alpha(k) = k$ for $k>4$, then by corollary 2.1 and Theorem 2.1, we can find $(F, +, \psi)$ in which $i_k = \alpha(k)$ for all k. Clearly, $i_3 = 2 \nmid 3$.

Remark 2.8 : A divisor d of a positive integer n is called au nitarydivisor if d and n/d are relatively prime, n is called a square-free if $p^2 \nmid n$ for any prime p. A divisor d of n is called a square-free unitary divisor of n, if d is both square-free and a unitary divisor of n. Let $\gamma^*(n)$ denote the largest square-free unitary divisor of n. Clearly, $\gamma^*(1) = 1$ and γ^* is multiplicative. Also for any prime p and any positive integer m, we have

$$\gamma^*(p^m) = \begin{cases} 1, \text{ if } m \geqslant 2 \\ p^m, \text{ if } m = 1 \end{cases} \qquad \text{.. (2.3)}$$

It is easy to see that $\gamma^*(m) \mid m$ and $\gamma^*(\gamma^*(m)) = \gamma^*(m)$, for all $m \in Z^+$. We can write any positive integer m uniquely as $m = \gamma^*(m)\, a^*(m)$, where $a^*(m) = m/\gamma^*(m)$. We note that $\gamma^*(m)$ and $a^*(m)$ are relatively prime and $p \mid a^*(m)$ implies that $p^2 \mid a^*(m)$ for any prime p. Hence from (2.1), it can be easily shown that $\gamma^*(a^*(m)\, a^*(n)) = 1$ for any $m, n \in Z^+$. Also, if $\gamma^*(m) = \gamma^*(n)$, then $\gamma^*(m) = \gamma^*([m, n])$. Let $T_* = \{(x, y) \in Z^+ \times Z^+ : \gamma^*(x) = \gamma^*(y)\}$, $\psi_1(x, y) = \max\{x, y\}$ and $\psi_2(x, y) = [x, y]$, for all $(x, y) \in T_*$. From Corollary 2.1 and Remark 2.2, we get that the triples $(F, +, \psi_1) = F\psi_1$ and $(F, +, \psi_2) = F_{\psi_2}$ are commutative rings whose unity is the characteristic function of $\{\gamma^*(k): k = 1, 2, \ldots\}$ (= The set of square-free integers) so that the unity is $|\mu|$. From Theorem 2.2 of Sitaramaiah[1] follows that $f \in F_{\psi_1}$ is invertible if and only if $\sum\limits_{\substack{x \leqslant k \\ \gamma^*(x) = \gamma^*(k)}} f(x) \neq 0$ for all k and $f \in F_{\psi_2}$ is invertible if and only if $\sum\limits_{\substack{x \mid k \\ \gamma^*(x) = \gamma^*(k)}} f(x) \neq 0$ for all k.

In the following example we shall define another function ψ on the domain T_*.

Example 2.1—We define $\psi : T_* \to Z^*$ by, $\psi(x, y) = xa^*(y)$.

If $\gamma^*(x) = \gamma^*(y)$, then
$$\begin{aligned}\psi(y, x) = ya^*(x) &= \gamma^*(y)\, a^*(y)\, a^*(x) \\ &= \gamma^*(x)\, a^*(x)\, a^*(y) \\ &= xa^*(y) \\ &= \psi(x, y).\end{aligned}$$

Clearly, $\psi(x, y) \geqslant \max\{x, y\}$, for all $(x, y) \in T_*$. Also, $\psi(1, 1) = 1$ and $\psi(\gamma^*(k), k) = \gamma^*(k)\, a^*(k) = k$, for all $k \in Z^+$.

We assume that $(y, z) \in T_*$ and $(x, \psi(y, z)) \in T_*$. So we have $\gamma^*(y) = \gamma^*(z)$ and $\gamma^*(x) = \gamma^*(\psi(y, z))$.

Also,

$$\begin{aligned}\gamma^*(\psi(y, z)) = \gamma^*(ya^*(z)) &= \gamma^*(\gamma^*(y)\, a^*(y)\, a^*(z)) \\ &= \gamma^*(\gamma^*(y)\, \gamma^*(a^*(y)\, a^*(z)) \\ &= \gamma^*(y),\end{aligned}$$

since $\gamma^*(y) = \gamma^*(z)$ implies that $a^*(y)\, a^*(z)$ is relatively prime to $\gamma^*(y)$. Thus we have $\gamma^*(x) = \gamma^*(y) = \gamma^*(z)$. Hence $(x, y) \in T_*$.

Also, since $\gamma^*(\psi(x, y)) = \gamma^*(x)$, $(\psi(x, y), z) \in T_*$. Thus $(x, y) \in T_*$ and $(\psi(x, y), z) \in T_*$. Similarly we can show that the statement" "$(x. y) \in T_*$ and $(\psi(x, y), z) \in T_*$" implies the statement "$(y, z) \in T_*$ and $(x, \psi(y, z)) \in T_*$".

If $(y, z) \in T_*$ and $(x, \psi(y, z)) \in T_*$, then by the definition of ψ we have

$$\begin{aligned}\psi(x, \psi(y, z)) &= \psi(\psi(y, z), x)\\ &= \psi(y, z)\, a^*(x)\\ &= ya^*(z)\, a^*(x)\\ &= xya^*(z)/\gamma^*(x).\end{aligned}$$

Similarly,

$$\begin{aligned}\psi(\psi(x, y), z) &= \psi(x, y)\, a^*(z)\\ &= xa^*(y)\, a^*(z)\\ &= xya^*(z)/\gamma^*(y)\\ &= xya^*(z)/\gamma^*(x).\end{aligned}$$

Since $\gamma^*(x) = \gamma^*(y)$.

Thus we have verified that ψ satisfies all the conditions (I) through (V) of Section 1. Hence by Theorem 2.2, we get that $(F, +, \psi)$ is a commutative ring whose unity is the charateristic function of the set $\{\gamma^*(k) : k = 1, 2, \ldots\}$ that is, $|\mu|$. Finally we note that for $(x, k) \in T_*$,

$$\begin{aligned}\psi(x, k) = k &\Leftrightarrow xa^*(k) = k\\ &\Leftrightarrow x = ka^*(k)\\ &\Leftrightarrow x = \gamma^*(k).\end{aligned}$$

Hence by Theorem 2.2 of Sitaramaiah[1], f is invertible in this ring if and only if, $f(\gamma^*(k)) \neq 0$ for all k or equivalently f vanishes nowhere on the set of square-free integers.

§3. Throughout the following we assume that the function ψ satisfies the condition (I) through (V) of Section 1. Suppose that the corresponding ring $(F, +, \psi)$ has an identity say g. If S_k and i_k are as given in (1.1) and (1.2), for $x \in S_{i_k}$, $(x, i_k) \in T =$ the domain of ψ, and $\psi(x, i_k) = i_k$. Since $(k, i_k) \in T$, by (III) of § 1, $(x, i_k) \in T$ and $(k, \psi(x, i_k)) = (k, i_k) \in T$ imply that $(x, k) \in T$ and $k = \psi(\psi(x, i_k) = \psi(x, (i_k, k))$ $= \psi(x, \psi(1_k\, k)) = (x, k)$, so that $x \in S_k$. Hence $i_k \leqslant x$. Since $\psi(x, i_k) = i_k$, by (V) of section 1, $x \leqslant i_k$ so that $x = i_k$. Hence $S_{i_k} = \{i_k\}$. So, by Lemma 2.1

$$1 = \sum_{x \in S_{i_k}} g(x) = g(i_k).$$

We wish to point out that $g(x)$ is an integer for any $x \in Z^+$. This is clear for $x = 1$ since $i_1 = 1$. We assume that $g(x)$ is an integer for $1 \leqslant x < r$. If $r = i_r$, then $g(r) = 1$. So, we may assume that $r \neq i_r$. Again by Lemma 2.1.

$$0 = \sum_{\psi(x, i_r) = r} g(x) = g(r) + \sum_{\substack{\psi(x, i_r) = r \\ x < r}} g(x)$$

from which it follows that $g(r)$ is an integer. Thus we have proved the following :

Theorem 3.1—If g is the unity of the ring $(F, +, \psi)$, then g is integer-valued and $g(i_k) = 1$ for all $k \in Z^*$ where i_k is defined by (1.2).

If $g(x) \geqslant 0$ for all $x \in Z^+$, we can prove the following :

Theorem 3.2—If g is the unity of the ring $(F, +, \psi)$ and $g(x) > 0$ for all $x \in Z^+$, then g is the characteristic function of the set S given in (1.3).

PROOF : We show that the equation $\psi(x, k) = n$ has a unique solution in S if $k = n$ and no solutions in S if $k \neq n$. It would then follow that the characteristic function X of S is the unity of $(F, +, \psi)$. Hence $g = X$. By Lemma 2.1 and Theorem 3.1

$$1 = \sum_{x \in S_k} g(x) = g(i_k) + \sum_{\substack{x \in S_k \\ x \neq i_k}} g(x) = 1 + \sum_{\substack{x \in S_k \\ x \neq i_k}} g(x),$$

so that $\sum_{\substack{x \in S_k \\ x \neq i_k}} g(x) = 0$. Since g is non negative, we conclude that

$$g(x) = 0 \text{ for all } x \in S_k \text{ and } x \neq i_k. \qquad \ldots(3.1)$$

If $\psi(i_r, k) = k$ for some $i_r \in S$, then $i_r \in S_k$ and since by Theorem 3.1, $g(i_r) = 1$, (3.1) implies that $i_r = i_k$. Hence the equation $\psi(x, k) = k$ has a unique solution in S. If $k \neq n$, by Lemma 2.1, $\sum_{\psi(x,k)=n} g(x) = 0$ and the non negativity of g implies that $g(x) = 0$ for all x such that $\psi(x, k) = n$. By Theorem 3.1, $g(x) = 1$ for $x \in S$. Hence the equation $\psi(x, k) = n$ has no solutions in S if $k \neq n$. Hence the theorem follows.

In all the examples given in section 3 of Sitaramaiah[1], the unity of each of the rings assumes the values 0 or 1. Below we give an example of a ring $(F, +, \psi)$ whose unity assumes negative values also.

Example 3.1—Let $1 < a_1 < a_2 < a_3$ be integers and $L = \{a_1, a_2, a_3\}$. We put $L_1 = L \times L$ and $L_2 = \{(k, k) : k \in Z^+, k \notin L\}$. Let $T = L_1 \cup L_2$. Clearly, $L_1 \cap L_2 = \phi$. We define $\psi : T \to Z^+$ as follows :

On L_2, we define $\psi(k, k) = k$; on L_1, we define ψ by

$$\psi(a_1, a_1) = a_1, \psi(a_2, a_2) = a_2, \psi(a_3, a_3) = a_3,$$

$$\psi(a_1, a_2) = \psi(a_1, a_3) = \psi(a_2, a_3) = a_3,$$

and

$$\psi(x, y) = \psi(y, x) \text{ for } (x, y) \in L_1.$$

It is clear that ψ satisfies the conditions (I), (III), (IV) and (V) of section 1. Only the verification of (II) is non-trivial. We observe that for $(y, z) \in T$, $(y, z) \in L_1$ if and only if $\psi(y, z) \in L$. Therefore, $(y, z) \in T$ and $(x, \psi(y, z)) \in T$ imply that either "$(y, z) \in L_1$ and $(x, \psi(y, z)) \in L_1$" or "$(y, z) \in L_2$ and $(x, \psi(y, z)) \in L_2$". From this observation, the verification of the equivalence of the statements "$(y, z) \in T$ and $(x, \psi(y, z)) \in T$" and "$(x, y) \in T$ and $(\psi(x, y), z) \in T$" is not difficult. We now show that $\psi(x, \psi(y, z)) = \psi(\psi(x, y), z)$, whenever $(y, z) \in T$ and $(x, \psi(y, z)) \in T$. If $(y, z) \in L_2$ and $(x, \psi(y, z)) \in L_2$ it is trivial to see that $\psi(x, \psi(y, z)) = \psi(\psi(x, y), z)$. So we assume that $(y, z) \in L_1$ and $(x, \psi(y, z)) \in L_1$. So, $\psi(y, z) \in L$ and $x \in L$. We distinguish the following cases :

Case 1 : $\psi(y, z) = a_1$. So, $y = z = a_1$. Therefore,

$$\psi(x, \psi(y, z)) = \psi(x, a_1) = \begin{cases} a_1, \text{ if } x = a_1 \\ a_3, \text{ if } x = a_2 \text{ or } a_3. \end{cases}$$

Also,

$$\psi(\psi(x, y), z) = \psi(\psi(x, a_1), a_1) = \begin{cases} a_1, \text{ if } x = a_1 \\ a_3, \text{ if } x = a_2 \text{ or } a_3. \end{cases}$$

Case 2 : $\psi(y, z) = a_2. \Rightarrow y = z = a_2$. We have

$$\psi(x, \psi(y, z)) = \psi(x, a_2) = \begin{cases} a_3, \text{ if } x = a_1 \text{ or } a_3 \\ a_2, \text{ if } x = a_2. \end{cases}$$

Also,

$$\psi(\psi(x, y), z) = \psi(\psi(x, a_2), a_2) = \begin{cases} a_3, \text{ if } x = a_1 \text{ or } a_3 \\ a_2, \text{ if } x = a_2. \end{cases}$$

Case 3 : $\psi(y, z) = a_3$. We have

$$\psi(x, \psi(y, z)) = \psi(x, a_3) = a_3.$$

If $z = a_3$, then $\psi(\psi(x, y), z) = \psi(\psi(x, y) = a_3$.

If $z = a_2$, then $\psi(y, z) = a_3$ implies that $y = a_1$ or a_3 and so,

$$\psi(\psi(x, y), z) = \psi(\psi(x, y), a_2) = a_3, \text{ since for}$$

$$y = a_1 \text{ or } a_3,\ \psi(x, y) = a_1 \text{ or } a_3.$$

If $z = a_1$, then $\psi(y, z) = a_3$ implies that $y = a_2$ or a_3 and so

$$\psi(x, y) = a_2 \text{ or } a_3. \text{ Hence}$$
$$\psi(\psi(x, y), z) = \psi(\psi(x, y)\, a_1) = a_3.$$

Thus in all the cases, $\psi(x, \psi(y, z)) = \psi(\psi(x, y), z)$ and hence ψ satisfies the condition (II) of section 1. Now, we define the arithmetical function g by

$$g(k) = \begin{cases} 1, & \text{if } k \neq a_3 \\ -1, & \text{if } k = a_3. \end{cases}$$

We wish to prove that g is the unity of the ring $(F, +, \psi)$ by using Lemma 2.1. By the definition of ψ, we have

$$S_k = \begin{cases} \{k\}, & \text{if } k \neq a_3 \\ \{a_1, a_2, a_3\}, & \text{if } k = a_3. \end{cases}$$

Therefore, for any $k \in Z^+$, by the definition of g,

$$\sum_{x \in S_k} g(x) = 1.$$

Let $k, n \in Z^+$ and $k \neq n$. If $k \neq a_1, a_2$ or a_3. $(x, k) \in T \Leftrightarrow (x, k) \in L_2 \Leftrightarrow x = k \Rightarrow \psi(x, k) = k$ and hence $\psi(x, k) = n$ has no solution at all.

Hence

$$\sum_{\psi(x,k)=n} g(x) = \text{empty sum} = 0$$

Suppose $k = a_1$ or a_2 or a_3. Then $(x, k) \in T \Leftrightarrow (x, k) \in L_1$.

$$\Leftrightarrow \psi(x, k) = a_1 \text{ or } a_2 \text{ or } a_3.$$

Hence $\psi(x, k) = n$ implies that $n = a_1$ or a_2 or a_3,

and $k \leqslant n$. Since $k \neq n$, we have $k < n$.

If $n = a_2$ and $k = a_1$, then

$$\sum_{\psi(x,k)=n} g(x) = \sum_{\substack{\psi(x,a_1)=a_2 \\ x \in L}} g(x) = \text{empty sum} = 0.$$

If $n = a_3$ and $k = a_1$, then

$$\sum_{\psi(x,k)=n} g(x) = \sum_{\substack{\psi(x,a_1)=a_3 \\ x \in L}} g(x) = g(a_2) + g(a_3) = 1 - 1 = 0$$

If $n = a_3$ and $k = a_2$, then

$$\sum_{\psi(x,k)=n} g(x) = \sum_{\substack{\psi(x,a_2)=a_3 \\ x \in L}} g(x) = g(a_1) + g(a_3) = 1 - 1 = 0.$$

Hence by Lemma 2.1, it follows that g is the unity of the ring $(F, +, \psi)$.

Reference

1. V. Sitaramaiah, *Indian J. pure appl. Math.* 16 (1985), 994-1008.

Indian J. pure appl. Math., 19 (1) : 11-16, January 1988

RESTRICTED GENERALIZED FROBENIUS PARTITIONS

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The present paper concerns a study of generalized Frobenius partitions with some restrictions on its parts. In particular, representations of the generating functions of these functions in terms of multidimensional theta functions and as sums of infinite products are obtained.

1. INTRODUCTION

A generalized Frobenius partition defined by Andrews[2] or simply an F-partition of a positive integer n is a two-rowed array of non-negative integers

$$\begin{pmatrix} a_1 & \ldots & a_r \\ b_1 & \ldots & b_r \end{pmatrix}$$

where each row is arranged in non-increasing order and $n = r + \sum_{i=1}^{r} (a_i + b_i)$. Anderws[2] considers the two functions $\phi_k(n)$ and $c\phi_k(n)$, where $\phi_k(n)$ is the number of those F-partitions of n in which the parts repeat at most k times and $c\phi_k(n)$ is the number of F-partitions of n in which the parts are distinct and are coloured with k given colours. We[4] have introduced the function $c\phi_{k,h}(n)$ which enumerates those F-partitions of n in which each part is repeated at most h times and is coloured with k given colours.

Let $a_{k,i}(n)$ $(i \leqslant k)$ denote the k-coloured (say, $r_1, \ldots, r_k$) F-partitions of n in which there are no odd parts of some i colours, say $r_1, \ldots, r_i$ in the top row and no even parts of $r_1, \ldots, r_i$ in the bottom row. Let $A_{k,i}(q)$ denote the generating function of $a_{k,i}(n)$ so that

$$A_{k,i}(q) = \sum_{n=0}^{\infty} a_{k,i}(n)\, q^n.$$

For example, the F-partitions enumerated by $a_{2,1}(1)$ are

$$\begin{pmatrix} 0_1 \\ 0_2 \end{pmatrix} \begin{pmatrix} 0_2 \\ 0_2 \end{pmatrix}$$

while those enumerated by $a_{2,1}(2)$ are

$$\begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix} \begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix} \begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix} \begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix} \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}$$

and

$$A_{2,1}(q) = 1 + 2q + 5q^2 + 8q^3 + 16q^4 + 26q^5 + \dots.$$

The F-partitions enumerated by $a_{3,1}(1)$ are

$$\begin{pmatrix} 0_1 \\ 0_2 \end{pmatrix}\begin{pmatrix} 0_1 \\ 0_3 \end{pmatrix}\begin{pmatrix} 0_2 \\ 0_2 \end{pmatrix}\begin{pmatrix} 0_3 \\ 0_3 \end{pmatrix}\begin{pmatrix} 0_3 \\ 0_2 \end{pmatrix}\begin{pmatrix} 0_2 \\ 0_3 \end{pmatrix}$$

while those enumerated by $a_{3,1}(2)$ are

$$\begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}\begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix}\begin{pmatrix} 0_3 \\ 1_1 \end{pmatrix}\begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix}\begin{pmatrix} 1_2 \\ 0_3 \end{pmatrix}\begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix}\begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}\begin{pmatrix} 0_3 \\ 1_2 \end{pmatrix}\begin{pmatrix} 1_3 \\ 0_2 \end{pmatrix}$$

$$\begin{pmatrix} 1_3 \\ 0_3 \end{pmatrix}\begin{pmatrix} 0_1 \\ 1_3 \end{pmatrix}\begin{pmatrix} 0_2 \\ 1_3 \end{pmatrix}\begin{pmatrix} 0_3 \\ 1_3 \end{pmatrix}\begin{pmatrix} 0_3\,0_2 \\ 0_3\,0_2 \end{pmatrix}\begin{pmatrix} 0_3\,0_1 \\ 0_3\,0_2 \end{pmatrix}\begin{pmatrix} 0_2\,0_1 \\ 0_3\,0_2 \end{pmatrix}$$

and

$$A_{3,1}(q) = 1 + 6q + 16q^2 + 42q^3 + \dots.$$

Similarly,

$$A_{3,2}(q) = 1 + 3q + 10q^2 + 18q^3 + \dots.$$

Andrews[2] has obtained the representations of the generating functions $\phi_k(q)$ of $\phi_k(n)$ and $C\phi_k(q)$ of $c\phi_k(n)$ in terms of multidimensional theta functions. The object of this paper is to obtain the analog of this result for the generating function $A_{k,l}(q)$ and also to indicate a procedure to express $A_{k,l}(q)$ as a sum of infinite products. Andrews[3] (Chapter 7) in fact suggests the study of generalised Frobenius partitions with restrictions on its parts. As an example he considers the function $a_{2,1}(n) = a(n)$ and obtains the representation (4) (obtained in Section 2 of this paper) for the generating function of $a(n)$. This has motivated us to generalise and obtain the results of this paper.

2. Representations of $A_{k,l}(q)$

Theorem 1—For $|q| < 1$,

$$A_{k,l}(q) = \frac{1}{(q)_\infty^{k-l}(q^2;q^2)_\infty^{l}} \sum_{m_1,\dots,m_{k-1}=-\infty}^{\infty} q^{Q'(m_1,\dots,m_{k-1})} \qquad \dots(1)$$

where for complex numbers a and q, $(a)_\infty = (a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n)$,

$(a)_k = (a)_\infty/(aq^k)_\infty$, for k any integer and $(a)_k = \prod_{n=0}^{k-1}(1 - aq^n)$ for a positive integer k and

$$Q'(m_1, \ldots, m_{k-1}) = \tfrac{1}{2} \sum_{j=1}^{k-i} (3m_j^2 + m_j) + 2m_{k-i+1}^2 + \ldots + 2m_{k-1}^2 + 2 \sum_{1 \leqslant j < j' \leqslant k-1} m_j m_{j'}. \qquad \ldots(2)$$

PROOF : From the General Principle of Andrews[2] it follows that $A_{k,i}(q)$ is the constant term in

$$\prod_{n=1}^{\infty} (1 + zq^n)^{k-i} (1 + zq^{2n-1})^i (1 + z^{-1}q^{n-1})^{k-i} (1 + z^{-1} q^{2n-1})^i$$

$$= \frac{1}{(q)_\infty^{k-i} (q^2; q^2)_\infty^{i}} \sum_{m_1, \ldots, m_k = -\infty}^{\infty} z^{m_1 + \cdots + m_k} q^{\binom{m_1+1}{2} + \cdots + \binom{m_{k-i}+1}{2} + m_{k-i+1}^2 + \cdots + m_k^2}$$

on using Jacobi's triple product identity

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2; q^2)_\infty (-qz; q^2)_\infty (-qz^{-1}; q^2)_\infty \qquad \ldots(3)$$

for $|q| < 1$ and $z \neq 0$. The rest of the proof is similar to the proof of Theorem 5.1 of Andrews[2].

Remark 1: When $i = k$, we get

$$A_{k,k}(q) = \frac{1}{(q^2; q^2)_\infty^k} \sum_{m_1, \ldots, m_{k-1} = -\infty}^{\infty} q^{2Q(m_1, \ldots, m_{k-1})},$$

where $Q(m_1, \ldots, m_{k-1}) = \sum_{i=1}^{k-1} m_i^2 + \sum_{1 \leqslant i < j \leqslant k-1} m_i m_j$. Using Theorem 5.2 of Andrews[2] we find $A_{k,k}(q) = C\phi_k(q^2)$.

Remark 2: From the definition of $a_{k,i}(n)$ it is clear that

$$a_{k,k-1}(n) < a_{k,k-2}(n) < \ldots < a_{k,1}(n) < c\phi_k(n).$$

Corollary 1—For $|q| < 1$,

$$A_{2,1}(q) = \frac{(q^3; q^3)_\infty}{(q)_\infty^2 (-q^3; q^3)_\infty} = \frac{\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2}}{(q)_\infty^2} \qquad \ldots(4)$$

$$A_{3,1}(q) = \frac{(q^3; q^3)_\infty (q^{15}; q^{15})_\infty}{(q)^2_\infty \, (q^2; q\,)_\infty}$$

$$\times [(-q^8; q^{15})_\infty (-q^7; q^{15})_\infty (-q^2; q^3)_\infty (-q; q^3)_\infty$$

$$+ q^2 (-q^{13}; q^{15})_\infty (-q^2; q^{15})_\infty (-q^4; q^3)_\infty (-q^{-1}; q^3)_\infty$$

$$+ q^7 (-q^{18}; q^{15})_\infty (-q^{-3}; q^{15})_\infty (-q^6; q^3)_\infty (-q^{-3}; q^3)_\infty]. \quad \ldots (5)$$

$$A_{3,2}(q) = \frac{(-q^2; q^2)^2_\infty (-q^4; q^4)_\infty}{(q)_\infty}$$

$$\times [(-q^5; q^8)_\infty \; (-q^3; q^8)_\infty \; (-q^2; q^4)^2_\infty$$

$$+ 2q^2 \; (-q^9; q^8)_\infty \; (-q^{-1}; q^8)_\infty \; (-q^4; q^4)^2_\infty]. \quad \ldots (6)$$

PROOF : From (1) we get

$$A_{2,1}(q) = \frac{1}{(q)_\infty \; (q^2; q^2)_\infty} \sum_{m_1=-\infty}^{\infty} q^{(3m_1^2 + m_1)/2}$$

$$= \frac{(q^3; q^3)_\infty \; (-q^2; q^3)_\infty \; (-q; q^3)_\infty}{(q)_\infty \; (q^2; q^2)_\infty} \text{, on using (3)}$$

$$= \frac{(q^3; q^3)_\infty}{(q)^2_\infty \; (-q^3; q^3)_\infty}.$$

The second part of (4) follows by observing that

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2} = (q^6; q^6)_\infty \; (q^3; q^6)^2_\infty$$

$$= (q^3; q^3)_\infty \; (q^3; q^6)_\infty$$

$$= \frac{(q^3; q^3)_\infty}{(-q^3; q^3)_\infty} \text{; using eqn. (1.2.5) (Andrews[1]).}$$

To prove (5), we have from (1)

$$A_{3,1}(q) = \frac{1}{(q)^2_\infty \; (q^2; q^2)_\infty} \sum_{m_1, m_2=-\infty}^{\infty} q^{(3m_1^2 + m_1 + 3m_2^2 + m_2 + 4m_1 m_2)/2}$$

(*equation continued on p.* 15)

$$= \frac{1}{(q)_\infty^2 \ (q^2; q^2)_\infty}$$

$$\times \left[\sum_{m_1=-\infty}^{\infty} q^{(15m_1^2+m_1)/2} \sum_{m_2=-\infty}^{\infty} q^{[3(m_2+2m_1)^2+m_2+2m_1]/2} \right.$$

$$+ q^2 \sum_{m_1=-\infty}^{\infty} q^{(15m_1^2+11m_1)/2} \sum_{m_2=-\infty}^{\infty} q^{[3(m_2+2m_1)^2+5(m_2+2m_1)]/2}$$

$$\left. + q^7 \sum_{m_1=-\infty}^{\infty} q^{(15m_1^2+21m_1)/2} \sum_{m_2=-\infty}^{\infty} q^{[3(m_2+2m_1)^2+9(m_2+2m_1)]/2} \right]$$

(after grouping separately terms with $m_1 \equiv 0, 1, 2 \pmod 3$ and rearranging).

If we now change $m_2 - 2m_1$ and use (3), we get (5).

To prove (6) we consider

$$A_{3,2}(q) = \frac{1}{(q)_\infty \ (q^2; q^2)_\infty^2} \sum_{m_1, m_2=-\infty}^{\infty} q^{(3m_1^2+m_1+4m_2^2+4m_1m_2)/2}$$

$$= \frac{1}{(q)_\infty \ (q^2; q^2)_\infty^2} \left[\sum_{m_1=-\infty}^{\infty} q^{4m_1^2+m_1} \sum_{m_2=-\infty}^{\infty} q^{2(m_2+m_1)^2} \right.$$

$$\left. + q^2 \sum_{m_1=-\infty}^{\infty} q^{4m_1^2+5m_1} \sum_{m_2=-\infty}^{\infty} q^{2(m_2+m_1)^2+2(m_2+m_1)} \right]$$

(after grouping separately terms with m_1 even and m_1 odd and rearranging). Changing m_2 m_2 to $m_2 - m_1$ and using (3), we obtain (6).

From (4) it immediately follows that

$$a_{2,1}(n) \equiv \begin{cases} 0 \pmod 2 & \text{if } n \text{ is odd} \\ p\left(\frac{n}{2}\right) \pmod 2 & \text{if } n \text{ is even.} \end{cases} \qquad \ldots(7)$$

Also from (4) we see that

'$a_{2,1}(n)$ is the number of ordinary partitions of n into two coloured parts (say red and blue) with no parts $\equiv 3 \pmod 6$ and no blue parts divisible by 6'. ...(8)

Again from (4) we find that

'$a_{2,1}(n)$ is the number of two coloured partitions in which no red part appears more than twice and no blue part is congruent to 3 (mod 6)'. ...(9)

It would be interesting to obtain combinatorial proofs of (7), (8) and (9).

In a recent paper we[4] have proved the following lemma.

Lemma—For $a > 0$, $a_1, \ldots, a_{k-1}$ integers and $|q| < 1$, the series

$$\sum_{n_1,\ldots,n_{k-1}=-\infty}^{\infty} q^{a\left(\sum_{i=1}^{k-1} n_i^2 + \sum_{1\leqslant i<j\leqslant k-1} n_i n_j\right) + \sum_{i=1}^{k-1} a_i n_i}$$

can be expressed as a sum of $2^{k-2}\,3^{k-3}\,4^{k-4}\ldots(k-1)$ infinite products.

Making use of this lemma we now prove the following theorem.

Theorem 2—For any k and i, $A_{k,i}(q)$ can be expressed as a sum of infinite products.

Since the proof of Theorem 2 is almost similar to the proof of the above lemma, we only give a sketch. If $i = k$, then $A_{k,k}(q) = C\phi_k(q)$ is a sum of infinite products by the lemma. The cases $k = 2$, $i = 1$ and $k = 3$, $i = 1$ or 2 have been considered in Corollary *1*. Now for $k > 3$ and $i < k$, first we group separately terms with $m_1, \ldots, m_{k-2}$ even and odd in the right hand side of (1) and use (3). Then we group terms with $m_1, \ldots, m_{k-3} \equiv 0,1,2 \pmod 3$ and again use (3). Proceeding like this we can express (in at most $k-2$ steps) the series on the right hand side of (1) as a sum of infinite products.

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References

1. G. E. Andrews, *The Theory of Partitions, Encyclopedia of Math and Its Applications*. Vol. 2, Addison-Wesley, Reading, 1976.
2. G. E. Andrews, *Math. Soc.* **49**, No. 301 (1984), 1-44.
3. G. E. Andrews, *q-Series, their Development and Application in Analysis Number Theory, Combinatorics, Physics and Computer Algebra*, Amer. Math. Soc. 1986.
4. Padmavathamma, *Fibonacci Quart.* (to appear)

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FIXED POINT THEORY AND ITERATION PROCEDURES

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Several general fixed point theorems are given for (quasi) metric spaces and normed linear spaces. The convergence of a sequence $\{x_n\}_{n=1}^{\infty}$, given by some iteration process, to a fixed point of a mapping T is the underlying theme.

1. INTRODUCTION

Let $T : X \to X$ where (X, d) is a complete metric space. This paper considers two questions.

(1) When does T have a fixed point?

(2) When does a sequence $\{x_n\}_{n=1}^{\infty}$, given by some iteration process, converge to a fixed point of T?

It will be shown that much of the theory for generalized contractions involves the following three conditions, where $\{x^*_{n+1}\}$ is also a sequence in X.

(a) $\lim d(x_n, x^*_{n+1}) = 0.$

(b) $\sum_{n=1}^{\infty} \frac{1}{n} d(x_n, x^*_{n+1}) < \infty.$

(c) $\sum_{n=1}^{\infty} d(x_n, x^*_{n+1}) < \infty.$

Some special cases we have in mind are : $x^*_{n+1} = x_{n+1}$, $x^*_{n+1} = Tx_n$, $x^*_{n+1} = x_{n+1} = Tx_n$, and $x^*_{n+1} = (1 - \alpha_n) x_n + \alpha_n Tx_n$. Even when $x^*_{n+1} = Tx_n$, one would not expect the conditions to give any information about the questions unless the iteration process which generates $\{x_n\}$ involves T in some way. The following two examples illustrate this.

Example 1—$Tx = x$ for x a real number. Then $\sum_{n=1}^{\infty} d(x_n, Tx_n) = 0$ for any sequence $\{x_n\}$. If $x_n = n$, the sequence $\{x_n\}$ does not converge, but every x is a fixed point of T.

Example 2 —$Tx = x + \frac{1}{x}$ for $x \in [1, \infty)$. Then $d(x_n, Tx_n) = \frac{1}{x_n}$. Let $x_n = n^2$ for $n \geqq 1$. Now $\sum_{n=1}^{\infty} d(x_n, Tx_n) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ but T has no fixed point. Also, $d(Tx, Ty) < d(x, y)$ for $x \neq y$.

Example 3—This example shows that even when the iteration process involves a continuous T, condition (c) may not imply T has a fixed point. Let B denote the unit ball in the sequence space c_0 and set $T((x_1, x_2, x_3, \ldots)) = (1, x_1, x_2, x_3, \ldots)$. $\|Tx - Ty\| = \|x - y\|$ and T has no fixed point. Let $x_{n+1} = \left(1 - \frac{1}{n^2}\right) x_n + \frac{1}{n^2} Tx_n$ for $n \geqq 1$.

Now $\|x_n - x_{n+1}\| = \frac{1}{n^2} \|x_n - Tx_n\| \leqq \frac{2}{n^2}$ implies $\sum_{n=1}^{\infty} \|x_n - x_{n+1}\| < \infty$.

Remarks : Assume T is continuous.

(1) If $x_n \to p = Tp$, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Moreover, this condition is not sufficient. That is, if $\lim_{n\to\infty} d(y_n, Ty_n) = 0$, T may not have a fixed point.

(2) If $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ with $\lim_{i\to\infty} x_{n_i} = q$, then $Tq = q$ if and only if $\lim_{n\to\infty} d(Tx_{n_i}, x_{n_i}) = 0$ [Hicks[3], Theorem 1].

(3) The following considers condition (c) when $x^*_{n+1} = x_{n+1} = T^n x$. If for some x in X, $\sum_{n=1}^{\infty} d(T^n x, T^{n+1} x) < \infty$ (or equivalently, there exists some $\phi : X \to [0, 1]$ such that $d(y, Ty) \leqq \phi(y) - \phi(Ty)$ for every $y \in \{Tx, T^2x, T^3x, \ldots\}$), then : $x_{n+1} = T^n x \to p = Tp$ and you have $d(T^n x, p) \leqq \phi(x)$ where $\phi(x)$ reduces to the Banach error estimate when T is a generalized contraction [Bollenbacher and Hicks[1], Theorem 3]. This includes many generalizations of Banach's fixed point theorem. See Bollenbacher and Hicks[1].

(4) T has a fixed point if and only if there exists x in X with $\sum_{n=0}^{\infty} d(T^n x, T^{n+1} x) < \infty$. The sufficiency follows from Remark 3 and the necessity is obvious.

Definitions—The following definitions will be needed in this paper. Suppose (X, τ) is a topological space. A function $f : X \to R$ is 'lower semicontinuous' if and only if for each real number a, $f^{-1}(a, \infty)$ is open in (X, τ). If x is a point in X and

$T : X \to X$, then the set $O(x, \infty) = \{x, Tx, T^2x, \ldots\}$ is called the orbit of x. $G : X \to [0, \infty)$ is T-orbitally lower semicontinuous at x if $\{x_n\}$ is a sequence in $O(x, \infty)$ and $\lim_{n\to\infty} x_n = x^*$ implies $G(X^*) \leqq \liminf_{n\to\infty} G(x_n)$.

If (X, d) is a metric space, a mapping $T : X \to X$ is called 'nonexpansive' if $d(Tx, Ty) \leqq d(x, y)$ for each $x, y \in X$. A mapping $T : X \to X$ is called 'quasinonexpansive' if p being a fixed point of T implies that $d(Tx, p) \leqq d(x, p)$ for all x in X. A normed linear space $(X, \|\cdot\|)$ is said to be 'strictly convex' if $\|\lambda x + (1 - \lambda) y\| \leqq 1$ for all λ, $0 < \lambda < 1$, and all $x, y \in X$ such that $\|x\| = \|y\| = 1$.

2. Results for Quasi-metric Spaces

A quasi-pseudo-metric for a set X is a function d from $X \times X$ to the non-negative real numbers such that for all x, y, and z in X, $d(x, z) \leqq d(x, y) + d(y, z)$ and $d(x, x) = 0$. A quasi-pseudo-metric d such that $x = y$ whenever $d(x, y) = 0$ is a quasi-metric. So in a quasi-metric space, we do not assume that $d(x, y) = d(y, x)$ for every x and y. Let (X, d) be a quasi-metric space. The family of all sets of the form $\{(x, y) : d(x, y) < \epsilon\}$, where $\epsilon > 0$, is a base for a quasi-uniformly U called the quasi-uniformity generated by d. A sequence $\{x_n\}_{n=1}^{\infty}$ in (X, U) is a 'Cauchy sequence' provided that for each $u \in U$ there exist a positive integer n_0 and a point x from X such that $\{x_n : n \geqq n_0\}$ is contained in $u[x] = \{y \in X : (x, y) \in u\}$. Therefore, a sequence $\{x_n\}_{n=1}^{\infty}$ in (X, d) is a Cauchy sequence provided that for each $\epsilon > 0$ there exist a positive integer n_0 and a point x from X such that $\{x_n : n \geqq n_0\}$ is contained in the set $\{y \in X : d(x, y) < \epsilon\}$. (X, d) is complete if every Cauchy sequence converges.

Theorem 1—Let (X, d) and(Y, ρ) be quasi-pseudo-metric spaces. Let $\{x_n\}_{n=0}^{\infty}$ and $\{x_n^*\}_{n=0}^{\infty}$ be two sequences in X. Suppose $f : X \to Y$. Then there exists $\phi : fX \to [0, \infty)$ such that $d(x_n, x_{n+1}^*) \leqq \phi(fx_n) - \phi(fx_{n+1})$ for $n = 0, 1, 2, \ldots$ if and only if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}^*)$ converges.

Proof : Suppose there exists $\phi : fX \to [0, \infty)$ such that $d(x_n, x_{n+1}^*) \leqq \phi(fx_n) - \phi(fx_{n+1})$ for $n = 0, 1, 2, \ldots$. For $k \geqq 0$,

$$S_k = \sum_{n=0}^{\infty} d(x_n, x_{n+1}^*) \leqq \sum_{n=0}^{k} [\phi(fx_n) - \phi(fx_{n+1})]$$

$$= [\phi(fx_0) - \phi(fx_1)] + [\phi fx_1 - \phi(fx_2)] + \ldots + [\phi(fx_k) - \phi fx_{k+1})]$$

$$= \phi(fx_0) - \phi(fx_{k+1}) \leqq \phi(fx_0).$$

The sequence $\{S_k\}_{k=0}^{\infty}$ of partial sums of the infinite series $\sum_{n=0}^{\infty} d(x_n, x^*_{n+1})$ is a nondecreasing sequence bounded above by $\phi(fx_0)$ and therefore convergent.

Suppose $\sum_{n=0}^{\infty} d(x_n, x^*_{n+1})$ converges. Define $\phi(fx_j) = \sum_{n=0}^{\infty} d(x_{j+n}, x^*_{j+n+1})$ for $j = 0, 1, 2, \ldots$ and $\phi(fx) = 0$ for all $x \in X$ such that $x \neq x_0, x_1, x_2, \ldots$. Let j be a nonnegative integer. Then

$$\sum_{n=0}^{k} d(x_{j+n}, x^*_{j+n+1}) - \sum_{n=0}^{k} d(x_{(j+1)+n}, x^*_{(j+n)+n+1})$$

$$= [d(x_j, x^*_{j+1}) + d(x_{j+1}, x^*_{j+2}) + \ldots + d(x_{j+k}, x^*_{j+k+1})]$$

$$- [d(x_{(j+1)}, x^*_{(j+1)+1}) + d(x_{(j+1)+1}, x^*_{(j+1)+2})$$

$$+ \ldots + d(x_{(j+1)+k}, x^*_{(j+1)+k+1})]$$

$$= d(x_j, x^*_{j+1}) - d(x_{(j+1)+k}, x^*_{(j+1)+k+1}).$$

Since $\{d(x_{(j+1)+k}, x^*_{(j+1)+k+1})\}_{k=0}^{\infty}$ is a subsequence of $\{d(x_k, x^*_{k+1})\}_{k=0}^{\infty}$ and $d(x_k, x^*_{k+1}) \to 0$ as $k \to \infty$ because $\sum_{n=0}^{\infty} d(x_n, x^*_{n+1})$ converges, then

$$d(x_{(j+1)+k}, x^*_{(j+1)+k+1}) \to 0 \text{ as } k \to \infty.$$

So

$$\lim_{k\to\infty} [\sum_{n=0}^{k} d(x_{j+n}, x^*_{j+n+1}) - \sum_{n=0}^{k} d(x_{(j+1)+n}, x^*_{(j+1)+n+1})]$$

$$= \lim_{k\to\infty} [d(x_j, x^*_{j+1}) - d(x_{(j+1)+k}, x^*_{(j+1)+k+1})] = d(x_j, x^*_{j+1}).$$

Thus

$$\phi(fx_j) - \phi(fx_{j+1}) = \sum_{n=0}^{\infty} d(x_{j+n}, x^*_{j+n+1}) - \sum_{n=0}^{\infty} d(x_{(j+1)+n}, x^*_{(j+1)+n+1})$$

$$= (x_j, x^*_{j+1}) \text{ for } j = 0, 1, 2, \ldots.$$

Theorem 2—Let (X, d) and (Y, ρ) be quasi-metric spaces with (X, d) complete. Suppose $f : X \to Y$ and $\phi : fX \to [0, \infty)$. If there exists $c > 0$ and a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that

$$\max\{d(x_n, x_{n+1}), c\,\rho(fx_n, fx_{n+1})\} \leq \phi(fx_n) - \phi(fx_{n+1})$$

for $n = 0, 1, 2, \ldots$, then :

(1) $\lim_{n\to\infty} x_n = p$ exists.

(2) $d(x_0, x_n) \leqq \phi(fx_0)$.

(3) If $y \to d(x_n, y)$ is continuous for $n = 0, 1, 2, \ldots$, then $d(x_n, p) \leqq \phi(fx_n)$ for $n = 0, 1, 2, \ldots$.

PROOF : To prove (1), we first observe that

$\max\{d(x_n, x_{n+1}), c\,\rho(fx_n, fx_{n+1})\} \leqq \phi(fx_n) - \phi(fx_{n+1})$ for $n = 0, 1, 2, \ldots$ implies that $d(x_n, x_{n+1}) \leqq \phi(fx_n) - \phi(fx_{n+1})$ for $n = 0, 1, 2, \ldots$. We know from Theorem 1 that $\sum_{k=0}^{\infty} d(x_k, x_{k+1})$ converges. So $\{\sum_{k=0}^{j} d(x_k, x_{k+1})\}_{j=0}^{\infty}$ is a Cauchy sequence.

Let $\epsilon > 0$. There exists a nonnegative integer n_0 such that $m > n \geqq n_0$ implies that $\sum_{k=n}^{m-1} d(x_k, x_{k+1}) < \epsilon$. Fix $n \geqq n_0$. Then we show that $\{x_m : m \geqq n\}$ is contained in the set $\{y \in X : d(x_n, y) < \epsilon\}$. Since $d(x_n, x_n) = 0 < \epsilon$, then x_n is in $\{y \in X : d(x_n, y) < \epsilon\}$. If $m > n$, then $d(x_n, x_m) \leqq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) = \sum_{k=n}^{m-1} d(x_k, x_{k+1}) < \epsilon$ and x_m is a member of the set $\{y \in X : d(x_n, y) < \epsilon\}$. So $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, d). X is complete gives (1).

To prove (2), let $n \geqq 1$. Then

$$d(x_0\, x_n) \leqq d(x_0, x_1) + d(x_1, x_2) + \ldots + d(x_{n-1}, x_n) = \sum_{k=0}^{n-1} d(x_k, x_{k+1})$$

$$\leqq \sum_{k=0}^{n-1} [\phi(fx_k) - \phi(fx_{k+1})] = \phi(fx_0) - \phi(fx_n) \leqq \phi(fx_0).$$

So $d(x_0, x_n) \leqq \phi(fx_0)$ holds for $n = 0, 1, 2, \ldots$

Assume $y \to d(x_n, y)$ is continuous for $n = 0, 1, 2, \ldots$ Then (1) and (2) imply that $d(x_0, p) = \lim_{n\to\infty} d(x_0, x_n) \leqq \lim_{n\to\infty} \phi(fx_0) = \phi(fx_0)$. Also, for $m > n$,

$$d(x_n, x_m) \leqq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leqq \sum_{k=n}^{m-1} [\phi(fx_k) - \phi(fx_{k+1})]$$

$$= \phi(fx_n) - \phi(fx_m) \leqq \phi(fx_n).$$

Since $x_m \to p$ as $m \to \infty$, then $d(x_n, p) = \lim_{n\to\infty} d(x_n, x_m) \leqq \lim_{n\to\infty} \phi(fx_n) = \phi(fx_n)$.

Remark 5 : Note that (1), (3), and (4) of the following theorem follow directly from Theorem 2 by setting $x_n = T^n x_0$. Also, (2) of Theorem 3 follows from the definition of T-orbital lower semicontinuity.

Theorem 3 [Hicks[2], Theorem 2]—Let (X, d) and $(Y\ \rho)$ be quasi-metric spaces with (X, d) complete. Suppose $T : X \to X, f : X \to Y$, and $\phi : fX \to [0, \infty)$. If there exists $x_0 \in X$ and $c > 0$ such that

$$\max \{d(y, Ty), c\, \rho\,(fy, fTy)\} \leqq \phi\,(fy) - \phi\,(fTy)$$

for all $y \in 0\,(x_0, \infty)$, then we have :

(1) $\lim\limits_{n\to\infty} T^n x_0 = x^*$ exists.

(2) $Tx^* = x^*$ if and only if $G(x) = d(x, Tx)$ is T-orbitally lower semicontinuous at x_0.

(3) $d\,(x_0, T^n x_0) \leqq \phi\,(fx_0)$.

(4) If $y \to d\,(z, y)$ is continuous for every $z \in 0\,(x_0, \infty)$, then

$$d\,(T^n x_0, x^*) \leqq \phi\,(fT^n x_0) \text{ and } d\,(x_0, x^*) \leqq \phi\,(fx_0).$$

Remark 6 : If X is metric, $X = Y$, $c = 1$, and $f = I$ in Theorem 3, the condition becomes $d\,(y, Ty) \leq \phi\,(y) - \phi\,(Ty)$. This clearly holds if $d\,(Ty, T^2 y) \leqq k\, d\,(y, Ty)$ or $d\,(Tu, Tv) \leq k\, d\,(u, v)$ for $0 < k < 1$. Just put $\phi\,(y) = \dfrac{1}{1-k} d\,(y, Ty)$.

3. Fixed Point Theorems for Banach Spaces

Theorem 4—Let $(X, \|\cdot\|)$ be a Banach space. Suppose $T : X \to X$, x_0 is an arbitrary point in X, and $x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n\, Tx_n$ for $n = 0, 1, 2, \ldots$, where $0 < a \leqq |\,\alpha_n\,|$. If there exists $\phi : X \to [0, \infty)$ such that $\|x_n - x_{n+1}\| \leqq \phi\,(x_n) - \phi\,(x_{n+1})$ for $n = 0, 1, 2, \ldots$, (or equivalently, $\sum\limits_{n=0}^{\infty} \|x_n - x_{n+1}\| < \infty$) then :

(1) $\lim\limits_{n\to\infty} x_n = p$ exists.

(2) $\|x_0 - x_n\| \leqq \phi\,(x_0)$ for $n = 0, 1, 2, \ldots$.

(3) $\|x_n - p\| \leqq \phi\,(x_n)$ for $n = 0, 1, 2, \ldots$.

(4) $Tp = p$ if and only if $G\,(p) \leqq \lim\limits_{n\to\infty} \inf G\,(x_n)$, where $G\,(x) = a\, \|x - Tx\|$ for each x in X.

Proof : If $(Y, \rho) = (X, \|.\|)$, $f = I$, $c = 1$, and $x_n = (1 - \alpha_{n-1})\, x_{n-1} + \alpha_{n-1}\, Tx_{n-1}$ for $n = 1, 2, \ldots$ in Theorem 2, then (1) and (2) follow from (1) and (2) in Theorem 2. Since X is a normed linear space, then $y \to \|x_n - y\|$ is continuous for every y in X. Thus, (3) follows from (3) in Theorem 2.

To prove (4), first assume that $Tp = p$. Then $G\,(p) = a\,\|p - Tp\| = 0 \leqq \lim\limits_{n\to\infty} \inf a\,\|x_n - Tx_n\| = \lim\limits_{n\to\infty} \inf G(x_n)$. Now assume that $G\,(p) \leqq \lim\limits_{n\to\infty} \inf G\,(x_n)$. We

know $G(x_n) = a\,\|x_n - Tx_n\| \leqq |\alpha_n|\,\|x_n - Tx_n\| = \|\alpha_n x_n - \alpha_n Tx_n\| = \|(1-1+\alpha_n)x_n - \alpha_n Tx_n\| = \|x_n - [(1-\alpha_n)x_n + \alpha_n Tx_n]\| = \|x_n - x_{n+1}\|$. Then $0 \leqq a\,\|p - Tp\| = G(p) \leqq \liminf_{n\to\infty} G(x_n) \leqq \liminf_{n\to\infty} \|x_n - x_{n+1}\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Note that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ since $\sum_{n=0}^{\infty} \|x_n - x_{n+1}\|$ converges. Since $a > 0$, then $a\,\|p - Tp\| = 0$ implies that $Tp = p$.

Remark 7 : The condition in (4) is essentially that G is lower semicontinuous on the set $\{x_0, x_1, x_2, \ldots\}$. This condition clearly holds if T is a continuous function. In particular, $G(p) \leqq \liminf_{n\to\infty} G(x_n)$ holds so that $Tp = p$ if T is nonexpansive.

Corollary 1—Let $T : X \to X$, where $(X\,\|.\|)$ is a Banach space. Let x_0 be a point in X and set $x_{n+1} = (1-\alpha)x_n + \alpha Tx_n$ for $n = 0, 1, 2, \ldots$, where $\alpha > 0$. Suppose $\sum_{n=0}^{\infty} \|x_n - Tx_n\|$ converges. Then there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$. $Tp = p$ if and only if $G(p) \leqq \liminf_{n\to\infty} G(x_n)$, where $G(x) = \alpha\,\|x - Tx\|$. Also, we have the error bounds from Theorem 4.

PROOF : The corollary follows from Theorem 4 since $(x_n - x_{n+1}) = \alpha(x_n - Tx_n)$.

Theorem 5—Let K be a closed, convex subset of a Banach space $(X, \|.\|)$. Let $T : K \to K$ be nonexpansive. For $n = 0, 1, 2, \ldots$, set $x_{n+1} = \alpha_0 x_n + \alpha_1 Tx_n + \alpha_2 T^2 x_n + \ldots + \alpha_k T^k x_n$, where $\alpha_i \geqq 0$, $\alpha_1 > 0$, and $\sum_{i=0}^{k} \alpha_i = 1$. If there exists x_0 in K such that $\sum_{n=0}^{\infty} \|x_n - x_{n+1}\|$ converges, then there exists x^* in K such that $\lim_{n\to\infty} x_n = x^*$ and $Tx^* = x^*$.

PROOF : The fact that there exists x^* in K such that $\lim_{n\to\infty} x_n = x^*$ follows from Theorem 1 and Theorem 2. It must now be shown that $Tx^* = x^*$. Since T is nonexpansive on K, then T is continuous on K and the mapping given by $\alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \ldots + \alpha_k T^k$ is continuous on K. It follows that

$$\begin{aligned} 0 &\leqq \|x^* - [\alpha_0 x^* + \alpha_1 Tx^* + \alpha_2 T^2 x^* + \ldots + \alpha_k T^k x^*]\| \\ &= \lim_{n\to\infty} \|x_n - [\alpha_0 x_n + \alpha_1 Tx_n + \alpha_2 T^2 x_n + \ldots + \alpha_k T^k x_n]\| \\ &= \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \end{aligned}$$

Then $\|x^* - [\alpha_0 x^* + \alpha_1 x^* + \alpha_2 T^2 x^* + \ldots + \alpha_k T^k x^*]\| = 0$ implies that $\alpha_0 x^* + \alpha_1 Tx^* + \alpha_2 T^2 x^* + \ldots + \alpha_k T^k x^* = x^*$. Then $Tx^* = x^*$ by Theorem 1 of Kirk[4].

Theorem 6—Let C be a closed, convex subset of a strictly convex Banch space $(X, \|.\|)$. Let $T : C \to C$ be a quasi-nonexpansive mapping. Set

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T x_n + \alpha_1 T x_a + \alpha_2 T^2 x_n + \ldots + \alpha_k T^k x_n \text{ for } n = 0, 1, 2, \ldots,$$

where $\alpha_i \geqq 0$, $\alpha_0 > 0$, $\alpha_1 > 0$, and $\sum_{i=0}^{k} \alpha_i = 1$. If there exists $x_0 \in C$ such that $\sum_{n=0}^{k} \|x_n - x_{n+1}\|$ converges, then there exists $x^* \in C$ such that $\lim_{n\to\infty} x_n = x_n$. $Tx^* = x^*$ if and only if $G(x^*) \leqq \lim_{n\to\infty} \inf G(x_n)$,

where

$$G(x) = \|x - [\alpha_0 x + \alpha_1 Tx + \alpha_2 T^2 x + \ldots + \alpha_k Tx^k x]\| \text{ for each } x \text{ in } C.$$

PROOF : We know from Theorem 1 and Theorem 2 that there exists x^* in C such that $\lim_{n\to\infty} x_n = x^*$. If $Tx^* = x^*$, then

$$G(x^*) = \|x^* - [\alpha_0 x^* + \alpha_1 T x^* + \alpha_2 T^2 x^* + \ldots + \alpha_k T^k x^*]\|$$

$$= \|x^* - \sum_{i=0}^{k} \alpha_i x^*\| = \|x^* - x^* \sum_{i=0}^{k} \alpha_i \| = \|x^* - x^*\|$$

$$= 0 \leqq \lim_{n\to\infty} \inf G(x_n).$$

If $G(x^*) \leq \lim_{n\to\infty} \inf G(x_n)$, then

$$0 \leqq \|x^* - [\alpha_0 x^* + \alpha_1 Tx^* + \alpha_2 T^2 x^* + \ldots + \alpha_k T^k x^*]\| = G(x^*)$$

$$\leqq \lim_{n\to\infty} \inf G(x_n) = \lim_{n\to\infty} \inf \|x_n - [\alpha_0 x_n + \alpha_1 Tx_n + \ldots + \alpha_k T^k x_n]\|$$

$$= \lim_{n\to\infty} \inf \|x_n - x_{n+1}\| = \lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

Then $x^* = (\alpha_0 x^* + \alpha_1 Tx^* + \alpha_2 T^2 x^* + \ldots + \alpha_k T^k x^*)$. By Lemma 1.5 of Singh and Nelson[6], we know that $(\alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \ldots + \alpha_k T^k) x^* = x^*$ implies that $Tx^* = x^*$.

The next theorem involves the Mann iteration process which was introduced by Mann[5] and which will now be described. Suppose $A = [a_{ij}]$ is an infinite matrix which satisfies :

(1) $a_{ij} \geqq 0$ for every i and j.

(2) $a_{ij} = 0$ for $j > i$.

(3) $\sum_{j=1}^{i} a_{ij} = 1$ for every i.

(4) $\lim_{i\to\infty} a_{ij} = 0$ for every j.

Suppose C is a convex subset of a linear space X and $T : C \to C$. Choose x_1 in C. Then the Mann iteration process $M(x_1, A, T)$ is defined $v_n = \sum_{k=1}^{n} a_{nk} x_k$ and $x_{n+1} = Tv_n$ for $n = 1, 2, \ldots$ Mann proved that if $(X, \|.\|)$ is a Banach space, C is closed as well as convex, and T is continuous, then the convergence of either $\{x_n\}_{n=1}^{\infty}$ or $\{v_n\}_{n=1}^{\infty}$ to y implies the convergence of the other to y, and their common limit y is a fixed point of T. If $\sum_{n=1}^{\infty} \|x_n - x_{n+1}\| < \infty$, $\{x_n\}$ is a Cauchy sequence and Mann's result implies that $x_n \to y$, $v_n \to y$, and $Ty = y$: We now look at a weaker condition when A is the Cesaro matrix.

Theorem 7—Let $(X, \|.\|)$ be a Banach space. Let C be a closed, convex subset of X. Suppose $T : C \to C$, $x_1 \in C$, and $\{x_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are the sequences in the Mann iteration process $M(x_1, A, T)$ where A is the Cesaro matrix. That is, $A = [a_{ij}]$ is the infinite matrix such that $a_{ij} = 0$ if $j > i$ and $a_{ij} = \frac{1}{i}$ if $j \leq i$. If $\sum_{n=1}^{\infty} \frac{1}{n} \|v_n - Tv_n\|$ converges, then their exists x^* in C such that $v_n \to x^*$ as $n \to \infty$. Furthermore, if T is continuous, then $x_n \to x^*$ as $n \to \infty$ and $Tx^* = x^*$.

PROOF : We know that $v_n = \sum_{k=1}^{n} a_{nk} x_k = \sum_{k=1}^{n} \frac{1}{n} x_k$. It follows that

$$\|v_n - v_{n+1}\| = \left\| \frac{1}{n} \sum_{k=1}^{n} x_k - \frac{1}{n+1} \sum_{k=1}^{n+1} x_k \right\|$$

$$= \left\| \frac{1}{n(n+1)} \sum_{k=1}^{n} x_k - \frac{1}{n+1} x_{n+1} \right\|$$

$$= \left\| \frac{1}{n+1} v_n - \frac{1}{n+1} Tv_n \right\| = \frac{1}{n+1} \|v_n - Tv_n\|$$

$$\leq \frac{1}{n} \|v_n - Tv_n\|. \text{ So } \sum_{n=1}^{\infty} \|v_n - v_{n+1}\|,$$

converges. Thus, there exists x^* in C such that $\lim_{n\to\infty} v_n = x^*$ by Theorem 1 and Theorem 2. If T is continuous, then we know that $\lim_{n\to\infty} x_n = x^*$ and $Tx^* = x^*$.

Application—Let $T : K \to K$ be defined by $Tz = iz$ for each z in K where $K = \{z : z \text{ is a complex number and } |z| \leq 1\}$ has the usual absolute value metric for complex numbers. Note that T is nonexpansive and thus continuous. T has a unique fixed point at 0. Let z_0 be a point in K such that $z_0 \neq 0$. We will show that functional

iteration cannot be used to find the fixed point but the iteration procedure given by $z_{n+1} = \frac{1}{2} z_n + \frac{1}{2} Tz_n$ for $n = 0, 1, \ldots$ can be used to locate the fixed point 0.

If we set $z_{n+1} = Tz_n$ for $n = 0, 1, \ldots,$ then $z_{n+1} = i^{n+1} z_0$. The sequence $\{z_n\}_{n=0}^{\infty}$ does not converge. Note that $\sum_{n=0}^{\infty} d(T^n z_0, T^{n+1} z_0) = \sum_{n=0}^{\infty} | i^n z_0 - i^{n+1} z_0 |$

$= \sum_{n=0}^{\infty} | z_0 | \sqrt{2}$ does not converge.

However, if $z_{n+1} = \frac{1}{2} z_n + \frac{1}{2} Tz_n$ for $n = 0, 1, \ldots,$ then

$$z_{n+1} = \tfrac{1}{2} z_n + \tfrac{1}{2} iz_n = \frac{1+i}{2} z_n = \left(\frac{1+i}{2}\right)^{n+1} z_0.$$

So

$$\sum_{n=0}^{\infty} d(z_n, z_{n+1}) = \sum_{n=0}^{\infty} \left| \left(\frac{1+i}{2}\right)^n z_0 - \left(\frac{1+i}{2}\right)^{n+1} z_0 \right|$$

$$= \sum_{n=0}^{\infty} \left| \left(\frac{1+i}{2}\right)^n \right| | z_0 | \left| 1 - \frac{1+i}{2} \right|$$

$$= \sum_{n=0}^{\infty} | z_0 | (\sqrt{2}/2)^{n+1} = (\sqrt{2} + 1) | z_0 | .$$

By Corollary 1, we know that $\{z_n\}_{n=0}^{\infty}$ must converge to the fixed point 0.

References

1. Alberta Bollenbacher, and T. L. Hicks, *Proc. Am. Math. Soc.* (to appear).
2. T. L. Hicks, *Math. Japonica*, (to appear).
3. L. Troy, Hicks, *Math. Japonica*, **22** (1978), 557–64.
4. W. A. Kirk, *Glasgow Math. J.* **12** (1971), 6–9.
5. W. Robert Mann, *Proc. Am. Math. Soc.* **4** (1953), 505–10.
6. L. Kanhaya Singh, and L. James Nelson, *Math. Japonica* **30** (1985), 963–70.

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ON ALMOST CONTACT FINSLER STRUCTURES ON VECTOR BUNDLE

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In the present paper almost contact Finsler structures on vector bundle has been defined and its integrability condition is obtained. The set of all transformations of almost contact Finsler connection is a abelian group under the product of mappings which is isomorphic to an additive group of tensor of type (1, 2). In this paper notations and terminology of Matsumoto[4] and Miron[6] are used.

INTRODUCTION

Let $V(M) = \{V, \pi, M\}$ be a vector bundle whose total space V is a $(n+m)$-dimensional C^∞-manifold and whose base space M is an n-dimensional C^∞-manifold. The projection map $\pi : V \to M$, $u \in V \to \pi(u) = x \in M$ where $u = (x, y)$, and $y \in R^m = \pi^{-1}(x)$ the fiber of $V(M)$ over x.

A non linear connection N on the total space V of $V(M)$ is a differentiable distribution, $N : u \in V \to N_u \in T_u(V)$ such that

$$T_u(V) = N_u \oplus V^v_u \qquad \ldots(1.1)$$

where

$$V^v_u = \{X \in T_u(V) : \pi_*(X) = 0\}. \qquad \ldots(1.2)$$

Now N_u is called the horizontal distribution and V^v the vertical distribution. Thus, $X \in T_u(V)$ can be decomposed as

$$X = X^H + X^V, \forall\ X^H \in N_u, X^V \in V^v_u. \qquad \ldots(1.3)$$

Let x^i, $i = 1, 2, \ldots, n$ and y^a, $a = 1, 2, \ldots, m$ be the coordinates of x and y such that (x^i, y^a) are the coordinates of $u \in V$. The local base of N_u is

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^a_i\,(x, y)\,\frac{\partial}{\partial y^a} \qquad \ldots(1.4)$$

and that of V^v_u is $\dfrac{\partial}{\partial y^a}$ where $N^a_i\,(x, y)$ are the coefficient of N. Their dual bases is $(dx^i, \delta y^a)$

where

$$\delta y^a = dy^a + N_i^a\ (x, y)\, dx^i. \qquad \ldots(1.5)$$

Let

$$X = X^i (x, y) \frac{\delta}{\delta x^i} + X^a (x, y) \frac{\partial}{\partial y^a},\ \forall\ X \in T_u (V).$$

Then

$$X^H = X^i (x, y) \frac{\delta}{\delta x^i},\ X^V = \widetilde{X}^a \frac{\partial}{\partial y^a},\ \widetilde{X}^a = X^a + N_i^a\ X^i. \qquad \ldots(1.6)$$

Let

ω be *a* 1-from

$$\omega = \omega_i (x, y)\, dx^i + \omega_a (x, y)\, dy^a.$$

Then

$$\omega^H = \widetilde{\omega}_i\, dx^i,\ \widetilde{\omega}_i = \omega_i - N_i^a\ (x, y)\, \omega_a;\ \omega^v = \omega_a\, \delta y^a \qquad \ldots(1.7)$$

which give that

$$\omega^H (X^v) = 0,\ \omega^v (X^H) = 0 \qquad \ldots(1.8)$$

where

$$\omega = \omega^H + \omega^v.$$

Also

$$(S \otimes U)^H = S^H \otimes U^H,\ (S + U)^H = S^H + U^H$$

$$(S \otimes U)^V = S^V \otimes U^V,\ (S + U)^V = S^V + U^V. \qquad \ldots(1.9)$$

The Finsler tensor field of Type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ on V has the following local form

$$T = T^{i_1, \ldots i_p,\ a_1, \ldots a_r}_{j_1, \ldots j_q,\ b_1, \ldots b_s}\ (x, y) \frac{\delta}{\delta x^{i_1}} \otimes \ldots \oplus \frac{\delta}{\delta x^{i_p}} \otimes dx^{a_1}$$

$$\ldots \otimes \ldots \otimes dx^{a_r} \otimes \frac{\partial}{\partial y}\ j_1 \otimes \ldots \frac{\partial}{\partial y}\ jq \otimes,\ \delta y^{b_1} \otimes \ldots \otimes \delta y^{b_s}. \qquad \ldots(1.10)$$

Definition 1.1—A Finsler connection on V is a linear connection $\nabla = F\Gamma$ on V with the property, that the horizontal linear space N_u, $u \in V$ of the distribution N are parallel, with respect to ∇ and the vertical spaces V_u^v, $u \in V$ are also parallel relative to ∇ (Miron[5]).

A linear connection ∇ on V is a Finsler connection on V if and only if :

$$(\nabla_X\, y^H)^V = 0;\ (\nabla_X\, y^V)^H = 0,\ \forall\ X, Y \in T_u (V). \qquad \ldots(1.11)$$

A linear connection ∇ on V is a Finsler connection on V if :

$$\nabla_X Y = (\nabla_X Y^H)^H + (\nabla_X Y^V)^V \;\forall\; X, Y \in T_u(V) \qquad \text{...(1.12a)}$$

$$\nabla_X \omega = (\nabla_X \omega^H)^H + (\nabla_X \omega^V)^V \;\forall\; \omega \in T_u^*(V) \text{ and } X \in T_u(V) \qquad \text{...(1.12b)}$$

for any Finsler connection we define

$$\nabla_X^H Y = \nabla_X X^Y ;\; \nabla_X^V Y = \nabla_X V^Y \;\forall\; X, Y \in T_u(V)$$

$$\nabla_X^H f = X^H f,\; \nabla_X^V f = X^V f \;\forall\; f \in \mathcal{F}(V) \qquad \text{...(1.13)}$$

where ∇^H is called the h-covariant derivative and ∇^V the V-covariant derivative.

Definition 1.2—Let $\omega \in T_u^*(V)$ be a differential q-form on V and ∇ is a linear connection on V. Then its exterior differential $d\omega$ is defined as :

$$(d\omega)(X_1,\dots X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} (\nabla_{X_i} \omega)(X_1,\dots \tilde{X}_i,\dots X_{q+1})$$

$$- \sum_{1\leqslant i<j\leqslant q+1} (-1)^{i+j} (T(X_i, X_j)(X_1,\dots\tilde{X}_i,\dots\tilde{X}_j$$

$$\dots X_{q+1}) \;\forall\; X_i \in T_u(V) \qquad \text{...(1.14)}$$

where T is the torsion tensor of ∇.

In particular for $q = 1$

$$d\omega(X, Y) = (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X) + \omega T(X, Y)$$

$$\forall X, Y \in T_u(V) \text{ and } \forall \omega \in T_u^*(V). \qquad \text{...(1.15)}$$

In the canonical coordinates (x^i, y^a), there exists a well determined set of differentiable functions on V.

$$F^i_{jk}(x, y),\; F^a_{bk}(x, y);\; C^i_{ja}(x, y),\; C^a_{bc}(x, y)$$

such that

$$\nabla_{\frac{\delta^H}{\delta x^k}} \frac{\delta}{\delta x^j} = F^i_{jk}(x, y) \frac{\delta}{\delta x^i} ;\; \nabla_{\frac{\delta^H}{\delta x^k}} \frac{\partial}{\partial y^b} = F^a_{bk}(x, y) \frac{\partial}{\partial y^a}$$

$$\nabla_{\frac{\partial^V}{\partial y^c}} \frac{\delta}{\delta x^j} = C^i_{jc}(x, y) \frac{\delta}{\delta x^i} ;\; \nabla_{\frac{\partial^V}{\partial y^c}} \frac{\partial}{\partial y^b} = C^a_{bc}(x, y) \frac{\partial}{\partial y^a}$$

where $F^i_{jk}(x, y)$ $F^a_{bk}(x, y)$ are called the coefficient of the h-connections ∇^H and $C^i_{jc}(x, y)$, $C^a_{bc}(x, y)$ are called the coefficients of of V-connections ∇^V.

The torsion tensor field T of a Finsler-connection is characterised by five. Finsler tensor fields.

$$[T(X^H, Y^H)]^H, [T(X^H, Y^H)]^V, [T(X^H, Y^V)]^H$$
$$[T(X^H, Y^V)]^V, [T(X^V, Y^V)]^V.$$

2. Almost Contact Finsler Structure on Vector Bundle

Let ϕ be an almost contact structure on V given by the tensor field of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with the property :

a. $\phi.\phi = -I_n + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V$

b. $\phi\, \xi^H = 0,\ \phi\xi^V = 0$

c. $\eta^H(\xi^H) + \eta^V(\xi^V) = 1$

d. $\eta^H(\phi X^H) = 0;\ \eta^V(\phi X^H) = 0,\ \eta^H(\phi X^V) = 0,\ \eta^V(\phi X^V) = 0$ ...(2.1)

where η is 1-form and ξ is vector field[1].

Proposition 2.1—If ϕ is an almost contact Finsler structure on V, there exists an unique decomposition of ϕ in the Finsler tensor fields,

$$\phi = \phi^1 + \phi^2 + \phi^3 + \phi^4 = \begin{pmatrix} \phi^1 & \phi^2 \\ \phi^3 & \phi^4 \end{pmatrix} = \begin{pmatrix} \phi^1 & \phi^2 \\ \phi^3 & \phi^4 \end{pmatrix} \qquad \text{...(2.2)}$$

where

$$\left.\begin{aligned} &\phi^1_a(\omega, X) = \phi(\omega^H, X^H),\ \phi^2(\omega, X) = \phi(\omega^H, X^V) \\ &\phi^3(\omega, X) = \phi(\omega^V, X^H)\ \phi^4(\omega, X) = \phi(\omega^V, X^V) \end{aligned}\right\} \qquad \text{...(2.3)}$$

$\forall\ X \in T_u(V)$ and $\forall\ \omega \in T^*_u\ (V)$.

We can write

$$\left.\begin{aligned} &\phi(X^H) = \phi^1(X^H) = \phi^3(X^H) \\ &\phi(X^V) = \phi^2(X^V) + \phi^4(X^V). \end{aligned}\right\} \qquad \text{...(2.4)}$$

It follows that

$$\left.\begin{aligned} &(\phi^1 \circ \phi^1 + \phi^2 \circ \phi^3)(X^H) = -X^H + \eta^H(X^H)\,\xi^H \\ &(\phi^3 \circ \phi^1 + \phi^4 \circ \phi^3) = 0 \\ &(\phi^1 \circ \phi^2 + \phi^2 \circ \phi^4)(X^V) = 0 \\ &(\phi^3 \circ \phi^2 + \phi^4 \circ \phi^4)(X^V) = -X^V + \eta^V(X)\,\xi^V. \end{aligned}\right\} \qquad \text{...(2.5)}$$

In terms of local coordinates ϕ is respresented by :

$$\phi = \phi^i_j \frac{\delta}{\delta x^i} \otimes dx^j + \phi^i_a \frac{\delta}{\delta x^i} \otimes \delta y^a + \phi^a_j \frac{\partial}{\partial y^a} \otimes dx^j + \phi^a_b \frac{\partial}{\partial y^a} \otimes \delta y^b$$

or

$$\phi = (\phi^A_B) = \begin{pmatrix} \phi^i_j & \phi^i_a \\ \phi^a_j & \phi^a_b \end{pmatrix}. \qquad \text{...(2.6)}$$

Then (2.5) takes the form

$$\left.\begin{aligned} \phi^i_j \ \phi^j_k + \phi^i_a \ \phi^a_k &= - \ \delta^i_k + \eta_k \otimes \xi^i \\ \phi^a_j \ \phi^j_k + \phi^a_b \ \phi^b_k &= 0 \\ \phi^i_j \ \phi^j_b + \phi^i_a \ \phi^a_b &= 0 \\ \phi^a_j \ \phi^j_b + \phi^a_c \ \phi^c_b &= - \ \delta^a_b + \eta_b \otimes \xi^a. \end{aligned}\right\} \qquad \text{...(2.7)}$$

The integrability tensor field of the almost contact Finsler structrure on V is given by[2]

$$\begin{aligned} \tilde{N}(X, Y) = [\phi X, \phi Y] - \phi [\phi X, Y] - \phi [X, \phi Y] + \phi^2 [X, Y] \\ + \ d\eta^H (X, Y) \xi^H + d\eta^V (X, Y) \xi^V, \ \forall \ X, Y \in T_u (V). \end{aligned}$$

$$\text{...(2.8)}$$

Proposition 2.2—If T is the torsion tensor of the almost contact Finsler connection ∇ then $\tilde{N}(X, Y)$ takes the form :

$$\tilde{N}(X, Y) = T(X, Y) + \phi T(X, \phi Y) + \phi T(\phi X, Y) - T(\phi X, \phi Y). \qquad \text{...(2.9)}$$

PROOF : We have

$$\begin{aligned} & T(X, Y) + \phi T(X, \phi Y) + (\phi T(\phi X, Y) - T(\phi X, \phi Y) \\ & = [\phi X, \phi Y] - \phi [X, \phi Y] - \phi [\phi X, Y] + \phi^2 [X, Y] \\ & \qquad + \phi^2 (T(X, Y)) + T(X, Y) \\ & = [\phi X, \phi Y] - \phi [X, \phi Y] - \phi [\phi X, Y] + \phi^2 [X, Y] \\ & \qquad + (\eta^H \otimes \xi^H + \eta^V \otimes \xi^V)(T(X, Y)) \end{aligned}$$

which gives (2.9) because of[2];

$$d\eta^H (X, Y) \xi^H + d\eta^V (X, Y) \xi^V = \{\eta^H \otimes \xi^H + \eta^V \otimes \xi^V\} (T(X, Y)).$$

Theorem 2.1—The almost contact finsler structure is integrable i.e. normal if and only if

$$T(X, Y) + T(\phi X, Y) + T(X, \phi Y) - T(\phi X, \phi Y) = 0 \qquad \ldots(2.10)$$

which takes the following form in the local cordinates system.

$$\left.\begin{aligned} T^h_{ij} + \phi^h_k (\phi^l_j T^k_{il} + \phi^l_i T^k_{lj}) - \phi^l_i \phi^k_j T^h_{lk} &= 0 \\ R^a_{ij} + \phi^a_k (\phi^l_j R^k_{il} + \phi^l_i R^k_{lj}) - \phi^l_i \phi^k_j R^a_{lk} &= 0 \end{aligned}\right\} \qquad \ldots(2.11)$$

$$\left.\begin{aligned} C^h_{ib} + \phi^h_k (\phi^l_b C^k_{il} + \phi^l_i C^k_{lb}) - \phi^l_i \phi^k_b C^h_{lk} &= 0 \\ P^a_{ib} + \phi^a_k (\phi^l_b P^k_{il} + \phi^l_i P^k_{lb}) - \phi^l_i \phi^k_b P^a_{lk} &= 0 \end{aligned}\right\} \qquad \ldots(2.12)$$

$$S^a_{bc} + \phi^a_k (\phi^l_c S^k_{bl} + \phi^l_b S^k_{lc}) - \phi^l_b \phi^k_c S^a_{lk} = 0 \qquad \ldots(2.13)$$

where

T^h_{ij}, R^a_{ij}, P^a_{ib}, C^h_{ib}, S^a_{bc} are the torsion tensor fields of the almost contact Finsler connection.

PROOF : The almost contact Finsler structure ϕ is integrable if and only if $\tilde{N}(X, Y) = 0$.

In local coordinate system $\tilde{N}(X, Y) = 0$ is equivalent to

$$\tilde{N}\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 0$$

$$\tilde{N}\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^a}\right) = 0$$

$$\tilde{N}\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right) = 0$$

which give (2.11), (2.12) and (2.13) respectively.

3. Set of all Almost Contact Finsler Connections

Definition 3.1—A linear connection ∇ on V with the properties[3]

(1) ∇ is an almost contact connection on V

(2) ∇ is Finsler connection relative to the distributions N and V^v i.e. $(\nabla_X Y^H)^V = 0$, $(\nabla_X Y^V)^H = 0$ is called an almost contact Finsler connection on V.

Theorem 3.1—The connection

$$\nabla_X = \tfrac{1}{2}\{(I_n + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V)\overset{\circ}{\nabla}_X - \phi \circ \overset{\circ}{\nabla}_X \circ \phi\}$$

$\forall\ X \in T_u(V)$, where $\overset{\circ}{\nabla}$ is an arbitrary fixed Finsler connection on V, is an almost contact Finsler connection.

PROOF : we have

$$\nabla_X Y = \tfrac{1}{2}\{(I_n + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V)\overset{\circ}{\nabla}_X Y - (\phi \circ \overset{\circ}{\nabla}_X \circ \phi)Y\}$$

$$= \tfrac{1}{2}\{\overset{\circ}{\nabla}_X Y + (\eta^H \otimes \xi^H + \eta^V \otimes \xi^V)\ \overset{\circ}{\nabla}_X Y - \phi \circ \phi\ \overset{\circ}{\nabla}_X Y\}$$

$$= \tfrac{1}{2}\{\overset{\circ}{\nabla}_X + (\eta^H \otimes \xi^H + \eta^V \otimes \xi^V)\ \overset{\circ}{\nabla}_X Y$$

$$+ \overset{\circ}{\nabla}_X Y - (\eta^H \otimes \xi^H + \eta^V \otimes \xi^V)\ \overset{\circ}{\nabla}_X Y\}$$

$$= \overset{\circ}{\nabla}_X Y.$$

Now

$$\nabla_X Y = \nabla_X (Y^H + Y^V) = \tfrac{1}{2}(I_n + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V)\ \dot{\nabla}_X$$

$$- \phi \circ \overset{\circ}{\nabla}_X \circ \phi\ (Y^H + Y^V)$$

which gives

$$\nabla_X Y^H = \tfrac{1}{2}\{(I_n + \eta^H \otimes \xi^H + \eta^V \xi^V)\ \overset{\circ}{\nabla}_X - \phi \circ \overset{\circ}{\nabla}_X . \phi\}\ Y^H$$

$$\nabla_X Y^V = \tfrac{1}{2}\{(I_n + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V)\ \overset{\circ}{\nabla}_X - \phi \circ \overset{\circ}{\nabla}_X \circ \phi\}\ Y^V.$$

But

$$[\tfrac{1}{2}(I_n + \eta^H \otimes \xi^H) + \eta^V \otimes \xi^V)\ \overset{\circ}{\nabla}_X - \phi \circ \nabla_X \circ \phi\}\ Y^H]^V = 0$$

$$[\tfrac{1}{2}(I_n + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V)\ \overset{\circ}{\nabla}_X - \phi \circ \nabla_X \circ \phi\}\ Y^V]^H = 0$$

$$(\nabla_X Y^H)^V = 0,\ (\nabla_X Y^V)^H = 0.$$

Theorem 3.2—If ∇ is an almost contact Finsler connection on V.,

Then

$\bar{\nabla}_X = \nabla_X + \Omega\ {}_A X \forall\ X \in T_u(V)^1$ is also an almost contact Finsler connection, where $A_X : T_u(V) \to T_u(V)$ is the Obata operator Ω of ϕ given by

$$\Omega A_X = \tfrac{1}{2}(I_n + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V)\ A_X - \phi\ A_X \phi$$

and A is a tensor field of type (1,2).

PROOF : We have

$$\bar{\nabla}_X Y = \nabla_X + \Omega\ A_X\ Y$$

(equation continued on p. 34)

$$\Rightarrow \bar{\nabla}_X Y^H + \bar{\nabla}_X Y^V = (\nabla_X Y^H + \Omega A_X Y^H) + (\nabla_X Y^V + \Omega A_X Y^V)$$

which gives

$$\bar{\nabla}_X Y^H = (\nabla_X + \Omega A_X) Y^H$$

$$\bar{\nabla}_X Y^V = (\nabla_X + \Omega A_X) Y^V.$$

Since ∇ is an almost contact Finsler connection :

$$(\nabla_X Y^H)^V = 0, (\nabla_X Y^V)^H = 0$$

Also

$$(A_X Y^H)^V = 0 \text{ and } (A_X Y^V)^H = 0.$$

Therefore,

$$(\bar{\nabla}_X Y^H)^V = 0, (\bar{\nabla}_X Y^V)^H = 0.$$

Hence, $\bar{\nabla}_X$ is an almost contact connection.

Theorem 3.3—There exists an almost contact Finsler connection, if and only if for any fixed Finsler connection $\bar{\nabla}$ on V.

$$[\Omega (\overset{\circ}{\nabla}_X + A_X) Y^H]^V = 0$$

$$[\Omega (\overset{\circ}{\nabla}_X + A_X) Y^V]^H = 0$$

where A is a tensor field of type (1,2) and $X, Y \in T_u(V)$.

PROOF : If we have A such that

$$[\Omega (\overset{\circ}{\nabla}_X + A_X) Y^H]^V = 0$$

$$[\Omega (\overset{\circ}{\nabla}_X + A_X) Y^V]^H = 0.$$

Then $\nabla_X = \Omega (\overset{\circ}{\nabla}_X + A_X)$ is an almost contact Finsler connection because

$$(\nabla_X Y^H)^V = 0, (\nabla_X Y^V)^H = 0$$

conversely; if ∇ is an almost contact Finnsler connection then for a fixed $\overset{\circ}{\nabla}$ there exists $A \in \tau^1_2(V)$ such that

$$\nabla_X = \Omega (\overset{\circ}{\nabla}_X + A_X) \ \forall \ X \in T_u(V)$$

since ∇_X being Finsler connection, hence

$$(\nabla_X Y^H)^V = 0, (\nabla_X Y^V)^H = 0$$

i.e.

$$[\Omega (\overset{\circ}{\nabla}_X + A_X) Y^H]^V = 0$$

$$[\Omega\,(\overset{\circ}{\nabla}_X + A_X)\, Y^V]^H = 0.$$

Theorem 3.4—The set of all transformation of almost contact Finsler connections forms an abelian group under the product of mappings which is isomorphic to the additive group of tensors of type (1,2).

PROOF : A mapping of two almost contact Finsler connection is given by

$$f_s : \bar{\nabla}_X = \nabla_X + \Omega\, A_s\, X,\; s \in I.$$

Let

$$f_1 : \bar{\nabla}_X = \nabla_X + \Omega A_1\, X$$

$$f_2 : \bar{\nabla}_X = \nabla_X + \Omega\, A_2\, X$$

be two mappings then the product of mappings is given by

$$f_2 f_1 : \bar{\nabla}_X = \nabla_X + \Omega\,(A_1 + A_2)\, X.$$

The identity of mapping is $f_0 : \bar{\nabla}_X = \nabla_X$ and inverse mapping of f_s is $-f_s$: $\bar{\nabla}_X = \nabla_X - \Omega\, A_s\,(X)$.

Therefore, set of all transformations of almost contact Finsler connections is an abelian group under the product of mappints.

Further, if ψ sends f_s to A_s then

$$\psi\,(f_s \circ f_t) = A_s + A_t = \psi\,(f_s) + \psi\,(f_t)$$

again for any A_s there exists a f_s such that

$$\psi\,(f_s) = A_s.$$

Thus, ψ is an isomorphism to an additive group of tensors of type (1,2).

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REFERENCES

1. G. H. Atanasiv, B. B. Sinha, and S. K. Singh, *Proc. Nat. Semi.* on Finsler Spaces. 3 (1984), 29-36.
2. G. H. Atanasiv, and E. Stoica, *Proc. Nat. Semi.* on Finsler Spaces 3 Barsov (9984) 37–48.
3. C. Francis, Klepp, *Proc. Nat. Semi.* on Finsler Spacess 3 Barsov (6984), 106-115.
4. M. Matsumoto, The theory of Finsler Connections. Publi. of the study gruop geometry, 5, Okayana Univ. 1970.
5. R. Miron, *Proc. Nat. Semi.* on Finsler Spaces. 2 Barsov (1982), 147–188.
6. R. Miron, Techniques of Finsler Geometry in the Theory of vector Bundles (To appear in Acta Math. Hung. 1983).

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ALGEBRAS WITH A UNITY COMMUTATOR*

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A pair of elements in a linear associative algebra (or a ring) is called a 'Heisenberg pair' if their commutator is unity. Necessary conditions or sufficient conditions are given for existence of a Heisenberg pair in various rings and algebras. Emphasis is on rings of endomorphisms on modules and on rings of matrices. The situation is completely analyzed for free modules. Consequences of an algebra containing a Heisenberg pair are developed. The paper lists some open questions and a conjecture.

1. INTRODUCTION

Questions concerning the existence and nature of linear operators whose commutator is a scalar multiple of the identity can be traced back to the early days of modern quantum mechanics[7]. Even in the special setting appropriate to quantum mechanics, the existence question was not satisfactorily answered until 1947[10]. Since then the study of commutators of linear operators on a Hilbert space has attracted considerable attention. (Halmos[5] and Putnam[8] discuss this in detail and each provide extensive references.) This paper addresses the collateral question of the nature of morphisms whose commutator is the identity. More specifically, if R is a ring and M is an R-module, then a pair α, β of R-endomorphisms on M is called a 'Heisenberg pair' if $\alpha\beta - \beta\alpha = I_M$. (Herein all rings have unity : modules are unital left modules ; subrings contain the same unity element as the ring ; and ring homomorphisms take unity into unity, unless otherwise noted.) Necessary and sufficient conditions for a free module to have a Heisenberg pair are given. It is useful—and of interest of itself—to consider Heisenberg pairs in the setting of algebras and rings. A ring (algebra) A is said to contain a Heisenberg pair if $a, b \in A$ such that the commutator $[a,b] = ab - ba$, is unity. This paper addresses the question of when a ring or algebra can or cannot contain a Heisenberg pair, with special consideration given to rings of endomorphisms or matrices. (Shoda[9] and Albert and Muckenhaupt[1] have considered commutators of matrices over fields.)

For a ring R, char R denotes the additive order of the unity of the ring ; $C(R)$ denotes the center of the ring; and $M_n(R)$ is the full ring of n by n matrices over R. If

*This paper is dedicated to Professor E. R. Keown.

M is an R-module, then $\mathrm{End}_R\, M$ is the ring of all R-endomorphisms on M and I_M is the identity mapping on M. An 'algebra' will mean a unital associative linear algebra over a commutative ring.

By way of explanation for the choice of the phrase "Heisenberg pair," it was picked up from correspondence and conversation with P. R. Halmos. He used it in the context of linear operators, but seeing as how it reveals the origins of this study, which now roams through rings, algebras, and modules, it seems appropriate to use it also in the more general context.

2. Heisenberg Pairs on Free Modules

A free module is said to have 'dimension' if every free basis of the module has the same cardinality; if that cardinal is n (finite or infinite), the module is said to have 'dimension' n. (See Leavitt.[6]) A ring R is said to be 'dimensional if every free R-module has dimension. (Cohn[2] uses the phrase 'invariant basis number.) If M is a free module which is not finitely generated, then M has dimension[4]. The class of dimensional rings includes all commutative rings[3] and all subrings of left or right Noetherian rings[2]. In this section conditions are given which guarantee the existence of a Heisenberg pair on a free module which has dimension.

Proposition 2.1—Let F be a free R-module. If either of the following hold, then there is a Heisenberg pair on F:

(1) F is not finitely generated, or

(2) F has finite dimension $n \geqslant 2$ and char R divides n.

Proof : Let B be a free basis for F. If B is infinite, write B as the disjoint union of sets $B_j = \{b_{ji} : i = 0; 2, \ldots\}$. Define functions α and β on B via :

$$\alpha b_{j0} = 0,\ \alpha b_{ji} = i b_{j,i-1}, \text{ for } i \geqslant 1.$$

$$\beta b_{ji} = b_{j,i+1}, \text{ for } i \geqslant 0.$$

Then $\alpha\beta - \beta\alpha = I_B$. Extend α and β to R-endomorphisms on F to obtain the desired Heisenberg pair.

If $|B| = n$, then write B as the disjoint union of sets $B_j = \{b_{ji} : i = 0, 1, \ldots, m-1\}$, where $m = \text{char } R$. Then define α and β via:

$$\alpha b_{j0} = 0,\ \alpha b_{ji} = i b_{j,i-1}, \text{ for } i = 1, \ldots, m-1.$$

$$\beta b_{ji} = b_{j,i+1}, \text{ for } i = 1, \ldots, m-2 \text{ and } \beta b_{j,m-1} = 0.$$

Then extend α and β to R-endomorphisms on F and the desired Heisenberg pair is obtained.

If char R does not divide the dimension of F in the finite dimensional case, the given α and β will not yield $\alpha\beta - \beta\alpha = 1$, since $(\alpha\beta - \beta\alpha)\, b_{j,m-1} = (1-m) b_{j,m-1}$.

Variations on the scheme given in the proof of Proposition 2. 1 yield many different Heisenberg pairs. In particular, if $\gamma \in \mathrm{End}_R F$ is invertible, then $\gamma\alpha\gamma^{-1}$, $\gamma\beta\gamma^{-1}$ is a Heisenberg pair if and only if α, β is.

As the next result shows, the dimension of F being a multiple of char R is necessary.

Proposition 2.2—Let R be a commutative ring and F a free R-module of finite dimension n. If there is a Heisenberg pair on F, then char R divides n.

Proof : Since $\mathrm{End}_R F$ is ring isomorphic to $M_n(R)$, there is a Heisenberg pair A, B in $M_n(F)$. However, the trace of $[A, B]$ is zero while the trace of $I \in M_n(F)$ is $n. I$, which is zero only if char R divides n.

Corollary 2.3—If R has characteristic zero, then a finite dimensional free module over R cannot have a Heisenberg pair.

The argument in Proposition 2. 2 realise on the trace property; tr AB = tr BA, which does not hold in general for matrices over noncommutative rings. In the next section a way around this obstacle will be found.

3. Heisenberg Pairs in Rings and Algebras

Proposition 3.1—Let A be an algebra over a commutative ring R such that A is a finite dimensional free module over R. If A contains a Heisenberg pair, then char R divides $\dim_R A$.

Proof : For each $a \in A$, define $\phi_a(x) = ax$, the left multiplication mapping induced by a. Then $a \to \phi_a$ gives an R-algebra monomorhism from A into $\mathrm{End}_R A$; this mapping takes $I \in A$ into I_A. The existence of a Heisenberg pair on A then yields a Heisenberg pair in $\mathrm{End}_R A$, which in turn forces : char R divides $\dim_R A$.

Proposition 3.2—Let R be a subring of a simple ring S with $0 < \dim_{C(S)} S = k < \infty$. If char S does not divide $n. k$, then $M_n(R)$ does not contain a Heisnberg pair.

Proof : As in the proof of Proposition 3.1, there is a ring monomorphism from S into $\mathrm{End}_{C(S)} S$ and hence into $M_k(C(S))$. Using this yields a ring monomorphism from $M_n(S)$ into $M_{nk}(C(S))$; unity goes into unity with each morphism. Since char $C(S)$ = char S does not divide $n. k$. there is no Heisenberg pair in $M_{nk}(C(S))$ and hence none in $M_n(S)$. Consequently, $M_n(R)$ cannot contain a Heisenberg pair.

Corollary 3.3—Let S be a simple ring which is finite dimensional over its center. If char S does not divide $\dim_{C(S)} S$, then S does not contain a Heisenberg pair.

Proposition 3.2 is of particular interest if S is a skewfield. One consequence of this and of Proposition 2.1 is the following extension of a venerable piece of linear algebra folklore.

Proposition 3.4—Let V be a vector space over a skewfield K such that $\dim_{C(K)} K = m < \infty$. Then the following are equivalent :

(a) There is a Heisenberg pair either in $\mathrm{End}_K V$ or in $M_m(K)$.

(b) Either (i) $\dim_K V = \infty$ or (ii) $\dim_K V = n < \infty$ and char K divides mn.

PROOF : The case where $\dim_K V = 1$ is trivial, so take $n > 1$ throughout the proof.

If there is a Heisenberg pair in $\mathrm{End}_K V$ and $\dim_K V = n < \infty$, then there is a Heisenberg pair in $M_n(K)$. The center of $M_n(K)$ is ring isomorphic to $C(K)$. Using $S = M_n(K)$ as the simple ring in Proposition 3.2 and $R = \{\lambda I_n : \lambda \in K\}$, since $\dim_{C(S)} S = n^3 m$, we have char S divides $n^3 m$. But char S = char K, which in this case is a prime, leading to char K divides either n or m.

If $M_m(K)$ contains a Heisenberg pair, then using $S = K = R$ in Proposition 3.2 yields char K divides m^2 and hence divides m.

If either $\dim_K V = \infty$ or $\dim_K V = n < \infty$ and char K divides n, then Proposi- 2.1 immediately gives a Heisenberg pair in $\mathrm{End}_K V$. If char K divides m, then free K-module K^m satisfies Proposition 2.1 (2) and hence $\mathrm{End}_K K^m \simeq M_m(K)$ contains a Heisenberg pair.

Given a ring (or algebra) which contains a Heisenberg pair, it is easy to form new which also have this property. Let Ω be the class of all rings (algebras) which contain a Heisenberg pair.

Then Ω is closed under the formation of :

(i) factor rings,

(ii) finite direct sums.

(iii) direct products,

(iv) full matrix rings.

Proposition 3.5—Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of R-algebras, where each is also a free R-module, R a commutative ring of characteristic zero, and f and g be epimorphisms. If B contains a Heisenberg pair, then each of A, B, C is infinite dimensional over R.

PROOF : Proposition 2.2 yields that, viewed as a free R-module, B must have infinite dimension. So A and C must be infinite dimensional over R.

Corollary 3.6—Let M be an infinite dimensional free R-module, where R is a commutative ring of characteristic zero. The R-algebra $\mathrm{End}_R M$ has no finite dimensional (non-zero) homomorphic image.

Immediately from Proposition 3.1 it is seen that if A is an algebra over a commutative ring R, char $R=0$, and A contains a Heisenberg pair, then $\dim_R A$ is infinite. In the case where R is a field, there is an easy alternate proof of this result which makes on use of trace.

ALTLRNATE PROOF : If $a, b \in A$ such that $[a, b] = 1$, then $[a^{n+1}, b] = (n+1)a^n$. Use this and induction on n to show $\{a_j : j = 0, 1, 2, \ldots, n\}$ is linearly independent for each n. (Both char $R = 0$ and R a field play a crucial role in this argument.)

4. QUESTIONS AND CONJECTURES

Question 4.1—For a given ring R, which R-modules will have a Heisenberg pair of R-endomorphisms?

For free modules this question was answered in Section 2. Considering the intimate connection between free modules and projective modules it does not seem unreasonable that similar results hold for projective modules. However, the obvious attack does not work.

Conjucture 4.2—If P is a projective R-module which is not finitely gererated, then there is a Heisenberg pair in $\mathrm{End}_R\, P$.

Another approach to answering Question 4.1 is to consider the exact sequence of R-modules : $F \to M \to 0$, where F is free, and look for a ring homomorphism from $\mathrm{End}_R\, F$ onto $\mathrm{End}_R\, M$, a situation which does not happen in general, but which can occur.

Question 4.3—In the setting above, when is $\mathrm{End}_R\, M$ a ring homomorphic image of $\mathrm{End}_R\, F$?

For skewfield K every K-module which is not finitely generated has a Heisenberg pair. It is natural to ask :

Question 4.4—Which rings have the property that every module which is not finitely generated has a Heisenberg pair?

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REFERENCES

1. A. Albert, and B. Muckenhaupt. *Michigan Math. J.* **4** (1957), 1.
2. P. M. Cohn, *Topology* 5 (1966), 215.
3. J. Dieudonne, *C. R. Acad. Sci. Paris* 215 (1942), 563.
4. T. Fujiwara, *Proc. Japan. Acad.* 31 (1955), 135.

5. P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer Verlag, New York, 1980.
6. W. G. Leavitt. *Trans. Amer. Math. Soc.* 103 (1962), 113.
7. J. von Neuman, *Mathematical Foundations of Quantum Mechanics*, Princeton University. Press. Princeton, 1955.
8. C. P. Putnam, *Commutation Properties of Hilbert Space Operators and Related Topics*, Springer-Verlag, Berlin 1967.
9. K. Shoda, *Japanese J. Math.* 13 (1936), 361.
10. A Winter, *Phys. Rev.* 71 (1947), 738.

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SEPARATION AXIOMS FOR BITOPOLOGICAL SPACES

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Recently, by using regularly open sets Jain[2] has defined and characterized some new separation axioms for topological spaces. In this paper we try to extend these separation axioms to bitopological spaces and study its relationships among themselves as well as with other known separation axioms.

INTRODUCTION

In a topological space X, a set A is said to be regularly open[5] if it is the interior of its own closure or, equivalently, if it is the interior of some closed set. The complement of a regularly open set is said to be regularly closed[5]. Veličko[8] has defined the set to be δ-open if it is expressible as a union of regularly open sets. The complement of a δ-open set is said to be δ-closed[8]. A point x in X is said to be a δ-adherent point of a set A of X if every regularly open set containing x has non-empty intersection with A. The set of all δ-adherent points of a set A is denoted by δ-cl A. A is δ-closed if and only if $A = \delta$-cl A.

The concept of bitopological spaces was introduced by Kelly[3]. A set equipped with two topologies is called a bitopological space. The purpose of the present paper is to extend separation axioms given by Jain[2] to bitopological spaces. Several properties of these new pairwise axioms are obtained.

1. PAIRWISE rT_0-SPACES

Definition 1.1[2]—A topological space X is said to be rT_0 if for any two distinct points of X, there exists a regularly open set containing one of the points but not the other, or equivaletly, there exists a δ-open set containing one of the points but not the other.

Definition 1.2—A space (X, T_1, T_2) is said to be pairwise rT_0 if for any two distinct points of X, there exists a set which is either T_1-regularly open or T_2-regularly

open containing one of the points but not the other, or equivalently, there exists either a $T_1-\delta$-open or a $T_2-\delta$-open set containing one of the points but not the other.

Obviously, every pairwise rT_0 space is pairwise T_0 (in the sense of Murdeshwar and Naimpally[6]). But the converse need not be true as following example shows.

Example 1.1—Let $X = \{a, b, c\}$, $T_1 = \{X, \phi, \{a\}, \{b, c\}\}$ and $T_2 = \{X, \phi, \{b\}\}$. Then, clearly, the space (X, T_1, T_2) is pairwise T_0 but not pairwise rT_0.

Theorem 1.1—A space (X, T_1, T_2) is pairwise rT_0 if and only if given two distinct points of X either their $T_1-\delta$-closures are distinct or their $T_2-\delta$-closures are distinct.

PROOF : Let (X, T_1, T_2) be a pairwise rT_0 space and let $x, y \in X$ be two distinct points. Suppose U is a $T_1-\delta$-open set containing x but not y. Then $y \in T_1-\delta\text{cl}\ \{y\} \subset X - U$ and so $x \notin T_1 - \delta\text{cl}\ \{y\}$. Hence $T_1 - \delta\text{cl}\ \{x\} \neq T_1 - \delta\text{cl}\ \{y\}$. Conversely, let x, y be two distinct points of X. Then either $T_1-\delta\text{cl}\ \{x\} \neq T_1-\delta\text{cl}\ \{y\}$ or $T_2 - \delta\text{cl}\ \{x\} \neq T_2 - \delta\text{cl}\ \{y\}$. In the former case let p be a point of X such that $p \in T_1 - \delta\text{cl}\ \{y\}$ and $p \notin T_1 - \delta\text{cl}\ \{x\}$. We assert that $y \notin T_1 - \delta\text{cl}\ \{x\}$. If $y \in T_1 - \delta\text{cl}\ \{x\}$ then $T_1-\delta\text{cl}\{y\} \subset T_1 - \delta\text{cl}\ \{x\}$, so that $p \in T_1 - \delta\text{cl}\ \{y\} \subset T_1-\delta\text{cl}\{x\}$. This contradicts the fact that $p \notin T_1 - \delta\text{cl}\ \{x\}$. Hence $y \notin T_1 - \delta\text{cl}\ \{x\}$. Thus $U = X - T_1 - \delta\text{cl}\ \{x\}$ is a $T_1 - \delta$-open set containing y but not x. The case $T_2 - \delta\text{cl}\ \{x\} \neq T_2 - \delta\text{cl}\ \{y\}$ can be dealt with similarly.

Theorem 1.2—A space (X, T_1, T_2) is pairwise rT_0 if either (X, T_1) or (X, T_2) is rT_0.

PROOF : Obvious.

The converse of Theorem 1.2 need not be true as can be seen from the following example.

Example 1.2—Let $X = \{a, b, c\}$, $T_1 = \{X, \phi, \{a\}, \{b, c\}\}$ and $T_2 = \{X, \phi, \{c\}, \{a, b\}\}$. Then, clearly the space (X, T_1, T_2) is pairwise rT_0 but neither (X, T_1) nor (X, T_2) is rT_0.

2. PAIRWISE rT_1-SPACES

Definition 2.1[2]—A space X is said to be rT_1 if whenever x and y are distinct points in X, there exists a regularly open set containing x but not y, or equivalently, there exists a δ-open set containining x but not y.

Definition 2.2—A space (X, T_1, T_2) is said to be weakly pairwise rT_1 if for every pair of distinct points x,y of X, there exists either a T_1-regularly open or a T_2-regularly open set containing x but not y, or equivalently, there exists a $T_1 - \delta$-open or $T_2 - \delta$-open set containing x but not y.

Obviously, every weakly pairwise rT_1 space is pairwise rT_0, but the converse need not be true as can be seen from the following example.

Example 2.1—Let $X = \{a, b, c\}$, $T_1 = \{X, \phi, \{a\}, \{b, c\}\}$ and $T_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then, clearly the space (X, T_1, T_2) is pairwise rT_0. It is not weakly pairwise rT_1. For, consider the pair of points b, c of X, we observe that any T_1-regularly open set containing b also contains c and any T_2-regularly open set containing c also contains b.

Definition 2.3[6]—A space (X, T_1, T_2) is said to be pairwise T_1 if for every pair of distinct points x and y of X, there exists a T_1-open set or a T_2-open set containing x but not y.

A pairwise T_1-space in the sense of Murdeshwar and Naimpally[6] will be called weakly pairwise T_1.

Theorem 2.1—Every weakly pairwise rT_1 bitopological space is weakly pairwise T_1.

Proof : Obvious.

The convese of the above Theorem 2.1 is not necessarily true.

Theorem 2.2—The following statements are equivalent :

(i) (X, T_1, T_2) is a weakly pairwise rT_1 space.

(ii) $T_1 - \delta\text{cl}\,\{x\} \cap T_2 - \delta\text{cl}\,\{x\} = \{x\}$ for every $x \in X$.

(iii) For every $x \in X$, the intersection of all $T_1 - \delta$-neighbourhoods and all $T_2 - \delta$-neighbourhoods of x is $\{x\}$.

Proof : (i) $\Rightarrow$ (ii)—Let $x \in X$ and $y \in T_1 - \delta\text{cl}\,\{x\} \cap T_2 - \delta\text{cl}\,\{x\}$ where $y \neq x$. Since X is weakly pairwise rT_1, therefore there is a $T_1 - \delta$-open set U such that $y \in U, x \notin U$ or there is a $T_2 - \delta$-open set V such that $y \in V, x \notin V$. In either case $y \notin T_1 - \delta\text{cl}\,\{x\} \cap T_2 - \delta\text{cl}\,\{x\}$. Hence $\{x\} = T_1 - \delta\text{cl}\,\{x\} \cap T_2 - \delta\text{cl}\,\{x\}$.

(ii) $\Rightarrow$ (iii)—If $x, y \in X$ such that $x \neq y$, then $x \notin T_1 - \delta\text{cl}\,\{y\} \cap T_2 - \delta\text{cl}\,\{y\}$, so there is a $T_1 - \delta$-open set or a $T_2 - \delta$-open set containing x but not y. Therefore, y does not belong to the intersection of all $T_1 - \delta$-neighbourhoods and all $T_2 - \delta$-neighbourhoods of x.

(iii) $\Rightarrow$ (i)—Let x and y be two distinct points of X. By hypothesis, y does not belong to a $T_1 - \delta$-neighbourhoods or a $T_2 - \delta$-neighbourhoods of x. Therefore there exists a $T_1 - \delta$-open set or a $T_2 - \delta$-open set containing x but not y. Hence X is weakly pairwise rT_1 space.

Definition 2.4—A space (X, T_1, T_2) is said to be pairwise rT_1 if for each pair of distinct points x, y of X, there exists a T_1-regularly open set containing x but not y and a T_2-regularly open set containing y but not x, or equivalently there exists a $T_1 - \delta$-open set containing x but not y and a $T_2 - \delta$-open set containing y but not x.

Obviously, every pairwise rT_1 space is pairwise T_1 (in the sense of Reilly[7]) but the converse may be false as is shown by the following example.

Example 2.2—Let R be the set of real numbers and T the co-countable topology. Then (X, T, T) is pairwise T_1 but it is not pairwise rT_1, because the only T-regularly open sets are ϕ and R.

Theorem 2.3—A space (X, T_1, T_2) is pairwise rT_1 if and only if (X, T_1) and (X, T_2) are rT_1.

PROOF : Let (X, T_1, T_2) be pairwise rT_1 space. Let x, y be two distinct points of X, then there exists a T_1-regularly open set U such that $x \in U, y \notin U$. Thus (X, T_1) is rT_1. Similarly, (X, T_2) is rT_1. Converse is obvious.

It follows from the preceding Theorem that every pairwise rT_1 space is weakly pairwise rT_1 and a pairwise rT_1 space is nothing but a bi-rT_1 space. We also note that a weakly pairwise rT_1 space need not be pairwise rT_1, which can be easily verified.

Corollary 2.1—A space (X, T_1, T_2) is pairwise rT_1 if and only if each singleton is both $T_1 - \delta$-closed and $T_2 - \delta$-closed.

PROOF : Obvious.

3. PAIRWISE rT_2-SPACES

Definition 3.1—A space (X, T_1, T_2) is said to be pairwise semi-rT_2 if for every pair of distinct points x, y of X, there exists a T_1-regularly open set U and a disjoint T_2-regularly open set V such that $x \in U, y \in V$, or $x \in V, y \in U$.

Obviously, every pairwise semi-rT_2 space is pairwise semi-T_2 (in the sense of Kim[4]). But the converse is false as is shown by the following example.

Example 3.1—Let $X = \{a, b\}$, T_1 is the discrete topology on X and $T_2=\{X,\phi\{a\}\}$. Then clearly the space (X, T_1, T_2) is pairwise semi-T_2 but not pairwise semi-rT_2.

Theorem 3.1—If (X, T_1, T_2) is pairwise semi-rT_2 then (X, T_1) and (X, T_2) are both rT_0 spaces.

PROOF : Obvious.

Corollary 3.1—Every pairwise semi-rT_2 space is weakly pairwise rT_1.

Definition 3.2—A space (X, T_1, T_2) is said to be pairwise rT_2 if for every pair of distinct points x, y of X, there exists a T_1-regularly open set U and a T_2-regularly open set V such that $x \in U, y \in V$ and $U \cap V = \phi$.

Theorem 3.2—Every pairwise rT_2 space is bi-rT_1.

PROOF : Obvious.

Corollary 3.2—Every pairwise rT_2 space is pairwise rT_1. The converse of Corollary 3.2 need not be true.

Definition 3.3[1]—A subset A of a space (X, T) is said to be N-closed relative to T every cover of A by regular open sets of X has a finite subcover.

Theorem 3.3—Let A be a subset of a pairwise rT_2 space (X, T_1, T_2) which is N-closed relative to T_1. Then A is $T_2 - \delta$-closed.

PROOF : If $A = X$, then A is obviously $T_2 - \delta$-closed. If $A \neq X$ then there is a point $x \in X - A$. Since X is pairwise rT_2, for each $y \in A$, there exists a T_2-regularly open set Uy and a T_1-regularly open set Vy such that $x \in Uy$, $y \in Vy$ and $Uy \cap Vy \neq \phi$. Then $\{Vy : y \in A\}$ is a T_1-regularly open cover of the set A which is N-closed relative to T_1, So it has a finite subcover say $Vy_1, Vy_2, \ldots, Vy_n$. Let $U = \bigcap_{i=1}^{n} Uy_i$, $V = \bigcup_{i=1}^{n} Vy_i$. Then U is $T_2 - \delta$-open, V is $T_1 - \delta$-open and $x \in U$, $A \subset V$ and $U \cap V = \phi$. Thus, $x \in U \subset X - A$, and so $X - A$ is $T_2 - \delta$-open. Hence A is $T_2 - \delta$-closed.

The implications between pairwise separation axioms are indicated by the following diagram :

$$\begin{array}{ccccc}
\text{Pairwise semi-}T_2 & \Rightarrow & \text{Weakly pairwise } T_1 & & \\
\Uparrow & & \Uparrow & & \\
\text{Pairwise semi-}rT_2 & \Rightarrow & \text{Weakly pairwise } rT_1 & & \\
\Uparrow & & \Uparrow & & \Downarrow \\
\text{Pairwise } rT_2 & \Rightarrow & \text{Pairwise } rT_1 & \Rightarrow & \text{Pairwise } rT_0 \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{Pairwise } T_2 & \Rightarrow & \text{Pairwise } T_1 & \Rightarrow & \text{Pairwise } T_0
\end{array}$$

4. PAIRWISE rR_0-SPACES

Definition 4.1—In a topological space (X, T) the δ-Kernal of a point x of X is the set δ Ker $\{x\} = \{y : x \in \delta\text{cl}\{y\}\}$; and the δ-Kernal of a subset A of X is the δ Ker $A = \cap \{U : U$ is δ-open and $A \subset U\}$.

Lemma 4.1—Let (X, T) be a topological space and let A be a subset of X, then δ Ker $A = \{x \in X : \delta\text{cl}\{x\} \cap A \neq \phi\}$.

PROOF : $x \notin \delta$ Ker A implies $x \notin \cap \{U : U$ is δ-open and $A \subset U\}$, so there is a δ-open set U such the $A \subset U$ and $x \notin U$. Therefore, $\delta\text{cl}\{x\} \cap U = \phi$ and $\delta\text{cl}\{x\} \cap A = \phi$. Now, $\delta\text{cl}\{x\} \cap A = \phi$, so $G = X - \delta\text{cl}\{x\}$ is a δ-open set such that $A \subset G$. Also, x does not belong to the intersection of all δ-open neighbourhoods of A, so $x \notin \delta$ Ker A.

Definition 4.2[6]—A space (X, T_1, T_2) is said to be pairwise R_0 if for every T_i-open set G, $x \in G$ implies $T_j - \text{cl}\{x\} \subset G$, $i, j = 1, 2$, $i \neq j$.

Definition 4.3—A space (X, T_1, T_2) is said to be pairwise rR_0 if for every $T_i - \delta$-open set G, $x \in G$ implies $T_j - \delta\text{cl}\{x\} \subset G$, $i, j = 1, 2$, $i \neq j$.

We give below two examples to show that pairwise rR_0 and pairwise R_0 are independent concepts.

Example 4.1—Let $X = \{a, b, c\}$, $T_1 = \{X, \phi, \{a, b\}, \{a, c\}, \{a\}\}$ and $T_2 = \{X, \phi, \{b, c\}, \{a, b\}, \{b\}\}$. Then (X, T_1, T_2) is pairwise rR_0 but not pairwise R_0.

Example 4.2—Let $X = \{a, b, c\}$, $T_1 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $T_2 = \{X, \phi, \{a, c\}, \{b, c\}, \{c\}\}$. Then clearly the space (X, T_1, T_2) is pairwise R_0 but not pairwise rR_0.

Theorem 4.1—A space (X, T_1, T_2) is pairwise rR_0 if and only if for each pair of distinct points x, y of X, $T_1 - \delta\text{cl}\{x\} \cap T_2 - \delta\text{cl}\{y\} = \phi$ or $\{x, y\} \subset T_1 - \delta\text{cl}\{x\} \cap T_2 - \delta\text{cl}\{y\}$.

PROOF : Let $T_1 - \delta\text{cl}\{x\} \cap T_2 - \delta\text{cl}\{y\} \neq \phi$ and $\{x, y\} \not\subset T_1 - \delta\text{cl}\{x\} \cap T_2 - \delta\text{cl}\{y\}$. Let $Z \in T_1 - \delta\text{cl}\{x\} \cap T_2 - \delta\text{cl}\{y\}$ and $x \notin T_1 - \delta\text{cl}\{x\} \cap T_2 - \delta\text{cl}\{y\}$. Then $x \notin T_2 - \delta\text{cl}\{y\}$ which implies that $x \in X - T_2 - \delta\text{cl}\{y\}$ which is $T_2 - \delta$-open set, but $T_1 - \delta\text{cl}\{x\} \not\subset X - T_2\text{-}\delta\text{cl}\{y\}$ because $Z \in T_2 - \delta\text{cl}\{y\}$, so the space $(X, T_1\, T_2)$ is not pairwise rR_0.

Conversely, let G be a $T_1 - \delta$-open set containing a point x of X. Suppose $T_2 - \delta\text{cl}\{x\} \not\subset G$, then there is a point $y \in T_2 - \delta\text{cl}\{x\}$ such that $y \notin G$ and $T_1 - \delta\text{cl}\{y\} \cap G = \phi$. Since $X - G$ is $T_1 - \delta$-closed and $y \in X - G$, hence $\{x, y\} \not\subset T_1 - \delta\text{cl}\{y\} \cap T_2 - \delta\text{cl}\{x\}$ and so $T_1 - \delta\text{cl}\{y\} \cap T_2 - \delta\text{cl}\{x\} \neq \phi$.

Theorem 4.2—The following statements are equivalent.

(a) (X, T_1, T_2) is a pairwise rR_0 space.

(b) For each $x \in X$, $T_i - \delta\text{cl}\{x\} \subset T_j - \delta\text{Ker}\{x\}$, $i, j = 1, 2$, $i \neq j$.

(c) For any $x, y \in X$, $y \in T_i - \delta\,\text{Ker}\{x\}$ if and only if $x \in T_j - \delta\,\text{Ker}\{y\}$, $i, j = 1, 2$, $i \neq j$.

(d) For any $x, y \in X$, $y \in T_i - \delta\text{cl}\{x\}$ if and only if $x \in T_j - \delta\text{cl}\{y\}$, $i, j = 1, 2$, $i \neq j$.

(e) For any $T_i - \delta$-closed set F and a point $x \notin F$, there exists a $T_j - \delta$-open set U such that $x \notin U$ and $F \subset U$, $i, j = 1, 2$, $i \neq j$.

(f) Each $T_i - \delta$-closed set F can be expressed as $F = \{G : \cap G \text{ is } T_j - \delta\text{-open and } F \subset G\}$, $i, j = 1, 2$, $i \neq j$.

(g) Each $T_i - \delta$-open set G can be expressed as the union of $T_j - \delta$-closed sets contained in G, $i, j = 1, 2$, $i \neq j$.

(h) For each $T_i - \delta$-closed set F, $x \notin F$ implies $T_j - \delta\text{cl}\{x\} \cap F = \phi$, $i, j = 1, 2$, $i \neq j$.

PROOF : (a) $\Rightarrow$ (b)—By Definition 4.1, for any $x \in X$ we have $T_j - \delta$ Ker $\{x\}$ $= \cap \{G : G$ is $T_j - \delta$-open and $x \in G\}$ and by Definition 4.3, each T_j - open set G containing x contains $T_i - \delta\text{cl}\ \{x\}$. Hence $T_i - \delta\text{cl}\ \{x\} \subset T_j - \delta$ Ker $\{x\}$, $i, j = 1, 2, i \neq j$.

(b) $\Rightarrow$ (c)—For any $x, y \in X$, if $y \in T_i - \delta$ Ker $\{x\}$ then $x \in T_i - \delta\text{cl}\ \{y\}$ and hence by (b) $x \in T_j - \delta$ Ker $\{y\}$.

(c) $\Rightarrow$ (d)—For any $x, y \in X$, if $y \in T_i - \delta\text{cl}\ \{x\}$ so $x \in T_i - \delta$ Ker $\{y\}$ and hence by (c) $y \in T_j - \delta$ Ker $\{x\}$ implies $x \in T_j - \delta\text{cl}\ \{y\}$.

(d) $\Rightarrow$ (e)—Let F be a $T_i - \delta$-closed set and a point $x \notin F$. Then for any $y \in F$, $T_i - \delta\text{cl}\ \{y\} \subset F$ and so $x \notin T_i - \delta\text{cl}\ \{y\}$. Now, by (d) $x \notin T_i - \delta\text{cl}\ \{y\}$ implies $y \notin T_j - \delta\text{cl}\ \{x\}$, that is there exists a $T_j - \delta$-open set Gy such that $y \in Gy$ and $x \notin Gy$. Let $G = \bigcup_{y \in F} \{Gy : Gy$ is $T_j - \delta$-open, $y \in Gy$ and $x \notin Gy\}$. Then G is a $T_j - \delta$-open set such that $x \notin G$ and $F \subset G$.

(e) $\Rightarrow$ (f)—Let F be a $T_i - \delta$-closed set and suppose that $H = \cap \{G : G$ is $T_j - \delta$-open and $F \subset G\}$. Then $F \subset H$ and it remains to show that $H \subset F$. Let $x \notin F$. Then by (e) there is a $T_j - \delta$-open set G such that $x \notin G$ and $F \subset G$ and hence $x \notin H$. Therefore, each $T_i - \delta$-closed set F can be written as $F = \cap \{G : G$ is $T_j - \delta$-open and $H \subset G\}$.

(f) $\Rightarrow$ (g)—Obvious.

(g) $\Rightarrow$ (h)—Let F be a $T_i - \delta$-closed set and $x \notin F$. Then $X - F = G$ (say) is a $T_i - \delta$-open set containing x. Then by (g), G can be written as the union of $T_j - \delta$-closed sets, and so there is a $T_j - \delta$-closed set H such that $x \in H \subset G$; and hence $T_j - \delta\text{cl}\ \{x\} \subset G$. Thus $T_j - \delta\text{cl}\ \{x\} \cap F = \phi$.

(h) $\Rightarrow$ (a)—Let G be a $T_i - \delta$-open set and $x \in G$. Then $x \notin X - G$ which is $T_i - \delta$-closed set, and by (h) $T_j - \delta\text{cl}\ \{x\} \cap (X - G) = \phi$, which implies that $T_j - \delta\text{cl}\ \{x\} \subset G$, $i, j = 1, 2, i \neq j$. Thus X is pairwise rR_0.

Definition 4.4—In a bitopological spaces (X, T_1, T_2), for any $x \in X$, we introduce the following notations :

(i) $\text{bi} - \delta\text{cl}\ \{x\} = T_1 - \delta\text{cl}\ \{x\} \cap T_2 - \delta\text{cl}\ \{x\}$

(ii) $\text{bi} - \delta\text{Ker}\ \{x\} = T_1 - \delta\text{Ker}\ \{x\} \cap T_2 - \delta\text{Ker}\ \{x\}$.

Theorem 4.3—In a pairwise rR_0 space (X, T_1, T_2), for any x and y in X, we have either $\text{bi} - \delta\text{cl}\ \{y\} = \text{bi} - \delta\text{cl}\ \{y\}$ or $\text{bi} - \delta\text{cl}\ \{x\} \cap \text{bi} - \delta\text{cl}\ \{y\} = \phi$.

PROOF : Suppose that $\text{bi} - \delta\text{cl}\{x\} \cap \text{bi} - \delta\text{cl}\ \{y\} \neq \phi$, and let $Z \in [T_1 - \delta\text{cl}\ \{x\} \cap T_2 - \delta\text{cl}\ \{x\}] \cap [T_1 - \delta\text{cl}\ \{y\} \cap T_2 - \delta\text{cl}\ \{y\}]$ then $T_1 - \delta\text{cl}\ \{z\} \subset T_1 - \delta\text{cl}\ \{x\} \cap T_1 - \delta\text{cl}\ \{y\}$ and $T_2 - \delta\text{cl}\ \{z\} \subset T_2 - \delta\text{cl}\ \{x\} \cap T_2 - \delta\text{cl}\ \{y\}$. Also $Z \in T_1 - \delta\text{cl}\ \{x\}$ which implies that $T_2 - \delta\text{cl}\ \{x\} \subset T_2 - \delta\text{cl}\ \{y\}$, this is so, because by (d) of the Theorem 4.2, $Z \in T_1 - \delta\text{cl}\ \{x\}$ then $x \in T_2 - \delta\text{cl}\ \{z\}$, implies $T_2 - \delta\text{cl}\ \{x\} \subset T_2 - \delta\text{cl}\{Z\}$

$\subset T_2 - \delta\text{cl}\,\{y\}$. Similarly, $Z \in T_2 - \delta\text{cl}\,\{x\}$ implies $T_1 - \delta\text{cl}\,\{x\} \subset T_1 - \delta\text{cl}\,\{y\}$ and $Z \in T_1 - \delta\text{cl}\,\{y\}$ implies $T_2 - \delta\text{cl}\,\{y\} \subset T_2 - \delta\text{cl}\,\{x\}$ and $Z \in T_2 - \delta\text{cl}\,\{y\}$ implies $T_1 - \delta\text{cl}\,\{y\} \subset T_1 - \delta\text{cl}\,\{x\}$, therefore $T_1 - \delta\text{cl}\,\{x\} \cap T_2 - \delta\text{cl}\,\{x\} \subset T_1 - \delta\text{cl}\,\{y\} \cap T_2 - \delta\text{cl}\,\{y\}$ and $T_1 - \delta\text{cl}\,\{y\} \cap T_2 - \delta\text{cl}\,\{y\} \subset T_1 - \delta\text{cl}\,\{x\} \cap T_2 - \delta\text{cl}\,\{x\}$. Hence $T_1 - \delta\text{cl}\,\{y\} \cap T_2 - \delta\text{cl}\,\{y\} = T_1 - \delta\text{cl}\,\{x\} \cap T_2 - \delta\text{cl}\,\{x\}$. This proves the Theorem.

Theorem 4.4—In a pairwise rR_0 space (X, T_1, T_2) for any x and y in X, we have either bi$-\delta$ Ker $\{x\}$ = bi$-\delta$Ker $\{y\}$ or bi$-\delta$Ker $\{x\} \cap$ bi$-\delta$Ker $\{y\} = \phi$.

PROOF : By virtue of statement (c) of Theorem 4.2, the proof is similar to that of Theorem 4.3.

Theorem 4.5—If a space (X, T_1, T_2) is pairwise rT_1, then it is pairwise rR_0.

PROOF : If (X, T_1, T_2) is pairwise rT_1, then by Theorem 2.3, (X, T_1) and (X, T_2) are both rT_1 spaces which implies that $T_1 - \delta\text{cl}\,\{x\} = \{x\} = T_2 - \delta\text{cl}\{x\}$. Thus (X, T_1, T_2) is pairwise rR_0.

It is easy to see that the converse of Theorem 4.5 need not be true in general.

Theorem 4.6—Every pairwise rT_0, pairwise rR_0 space is weakly pairwise rT_1.

PROOF : Let (X, T_1, T_2) be pairwise rT_0 and pairwise rR_0 space. Let x and y be any pair of distinct points of X. Therefore, there exists a $T_i - \delta$-open set G, $i = 1$ or 2, containing one of the points but not the other. Suppose G contaınts x (say). Since X is pairwise rR_0, $x \in G$ implies $T_j - \delta\text{cl}\,\{x\} \subset G$, $j = 1, 2$ and $j \neq i$. Now $x \notin X - T_j - \delta\text{cl}\,\{x\} = M$ (say). Thus G is a $T_i - \delta$-open set containing x but not y and M is a $T_j - \delta$-open set containing y but not x. Thus x is weakly pairwise rT_1.

5. PAIRWISE rR_1-SPACES

Definition 5.1—A space (X, T_1, T_2) is said to be pairwise rR_1, if for every pair of distinct points x and y of X such that $T_i - \delta\text{cl}\,\{x\} \neq T_i - \delta\text{cl}\,\{y\}$, there exists a $T_j - \delta$-open set U and a $T_i - \delta$-open set V such that $x \in U$, $y \in V$ and $U \cap V = \phi$, $i, j = 1, 2, i \neq j$.

Theorem 5.1—Every pairwise rR_1 space is pairwise rR_0.

PROOF : Let (X, T_1, T_2) be pairwise rR_1. Let G be any $T_i - \delta$-open set and $x \in G$. For each point $y \in X - G$, $T_j - \delta\text{cl}\,\{x\} \neq T_i - \delta\text{cl}\,\{y\}$, so there exists a $T_i - \delta$-open set Uy and a $T_j - \delta$-open set Vy such that $x \in Uy$, $y \in Vy$ and $Uy \cap Vy = \phi$. If $A = \cup\,\{Vy : y \in X - G\}$, then $X - G \subset A$ and $x \notin A$. $T_j - \delta$-openness of A implies $T_j - \delta\text{cl}\,\{x\} \subset X - A \subset G$. Hence X is pairwise rR_0.

Theorem 5.2—A space (X, T_1, T_2) is pairwise rR_1 if and only if for every pair of points x and y of X such that $T_i - \delta\text{cl}\,\{x\} \neq T_j - \delta\text{cl}\,\{y\}$, there exists a $T_i - \delta$-open set U and a $T_j - \delta$-open set V such that $T_i - \delta\text{cl}\,\{x\} \subset V$, $T_j - \delta\text{cl}\,\{y\} \subset U$ and $U \cap V = \phi$, $i, j = 1, 2, i \neq j$.

PROOF : Let (X, T_1, T_2) be pairwise rR_1 space. Let x, y be points of X such that $T_i - \delta\text{cl}\{x\} \neq T_j - \delta\text{cl}\{y\}, i, j = 1, 2, i \neq j$. Then there exists a $T_i - \delta$-open set U and a $T_j - \delta$-open set V such that $x \in V, y \in U$ and $U \cap V = \phi$. Since a pairwise rR_1 space is pairwise rR_0, therefore, $x \in V$ implies $T_i - \delta\text{cl}\{x\} \subset V$ and $y \in U$ implies $T_j - \delta\text{cl}\{y\} \subset U$, $i, j, = 1, 2, i \neq j$. Hence the result follows. The converse is obvious.

Theorem 5.3—Every pairwise rT_2 space is pairwise rR_1.

PROOF : Obvious.

The converse of Theorem 5.3 is not true. However, we have the following result.

Theorem 5.4—Every pairwise rT_1, pairwise rR_1 space is pairwise rR_2.

PROOF : Let (X, T_1, T_2) be pairwise rT_1 and pairwise rR_1. Let x, y be two distinct points of X. Since the pairwise rT_1 space is bi$-rT_1$, $\{x\}$ is $T_2 - \delta$-closed and $\{y\}$ is $T_1 - \delta$-closed. Hence $T_2 - \delta\text{cl}\{x\} \neq T_1 - \delta\text{cl}\{y\}$. Since X is pairwise rR_1, there exists a $T_1 - \delta$-open set U and a $T_2 - \delta$-open set V such that $x \in U, y \in V$ and $U \cap V = \phi$. Hence X is pairwise rT_2.

Corollary 5.1—A space $(X, T_1 T_2)$ is pairwise rT_2 if and only if it is pairwise rT_1 and pairwise rR_1.

Theorem 5.5—Let A be a subset of a pairwise rR_1 space (X, T_1, T_2) which is N-closed relative to T_2 such that $A \cap T_2 - \delta\text{cl}\{x\} = \phi$ for some $x \in X$. Then there exists a $T_1 - \delta$-open set U and a $T_2 - \delta$-open set V such that $T_2 - \delta\text{cl}\{x\} \subset U, A \subset V$ and $U \cap V = \phi$.

PROOF : For each $y \in A$, $T_1 - \delta\text{cl}\{y\} \neq T_2 - \delta\text{cl}\{x\}$. Since X is pairwise rR_1, there exists a $T_1 - \delta$-open set Uy and a $T_2 - \delta$-open set Vy such that $T_1 - \delta\text{cl}\{y\} \subset Vy$, $T_2 - \delta\text{cl}\{x\} \subset Uy$ and $Uy \cap Vy = \phi$. But $\{Vy : y \in A\}$ is a $T_2 - \delta$-open cover of a set A which is N-closed relative to T_2 and hence admits a finite subcover, say $\{Vy_i : i = 1, 2,..., n\}$. Let $U = \bigcap_{i=1}^{n} Uy_i$, $V = \bigcup_{i=1}^{n} Vy_i$. Then U is $T_1 - \delta$-open, V is $T_2 - \delta$-open and $A \subset V$, $T_2 - \delta\text{cl}\{x\} \subset U$, $U \cap V = \phi$.

REFERENCES

1. D. Carnahan, *Boll. Un. Mat. Ital.* (4) 6, (1972), 146-53.
2. R. C. Jain, Ph. D. Thesis, Meerut University, Institute of Advanced Studies, 1980.
3. J. C. Kelly, *Proc. London Math. Soc.* (3) 13 (1963), 71-89.
4. Y. W. Kim, *Proc. Japan Acad.* 44 (1968), 1009-12.
5. C. Kuratowski, *Topologie I*, 4th edn., in French, Hanfer, New York, 1958.
6. G. M. Murdeshwar, and S. A. Naimpally, *Quasi-Uniform Topological Spaces*. Monograph P. Noordhroff Ltd., 1966.
7. I. L. Reilly, *Nanta Math.* (2) 5 (1972), 14-25.
8. N. V. Veličko, *Mat. Sb*, 70 (1966), 98-112.

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ON CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS

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The object of the present paper is to derive a result concerning Bazilevič functions of type β and order ρ defined in the unit disk. As special cases of our result, some results on the classes of convex functions of order β or of α-starlike functions are obtained.

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $E = \{z: |z| < 1\}$. Further let S be the subclass of A consisting of those functions that are univalent in the unit disk E.

A function $f(z)$ in A is said to be starlike if and only if it satisfies

$$\text{Re } \frac{zf'(z)}{f()} > 0$$

for all $z \in E$. We shall denote by S^* the subclass of A consisting of functions which are starlike in E.

Note that $S^* \subset S$, where S^* is the set of univalent mappings for which $f(E)$ is starlike with respect to the origin.

We can easily verify that if $f(z) \in A$ and $-1/2 \leqq \alpha < 1$, then

$$1 + \text{Re } \frac{zf''(z)}{f'(z)} > \alpha \qquad (z \in E)$$

implies the univalence of $f(z)$ (Ozaki[7], Umezawa[11] or Pfaltzgraff *et al.*[8]).

A function $f(z)$ in A is said to be close-to-convex of order α if there exists a funtion $g(z) \in S^*$ such that

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > \alpha$$

for all $z \in E$, for some α $(0 \leqq \alpha < 1)$. We denote by $K(\alpha)$ the subclass of A consisting of all functions which are close-to-convex of order α.

The class $K(0)$ when $\alpha = 0$ was introduced by Ozaki[6] and Kaplan[3].

A function $f(z)$ in A is said to be a Bazilevič function of type β and order ρ if there exists a function $g(z) \in S^*$ such that

$$\operatorname{Re} \frac{zf'(z)}{f(z)^{1-\beta} g(z)^{\beta}} > \rho \qquad \ldots(1.1)$$

for all $z \in E$, for some β $(0 < \beta)$, ρ $(0 \leqq \rho < 1)$. We denote by $B(\beta, \rho)$ the class of all such functions.

Thomas[10] called a function $f(z)$ satisfying (1.1) a Bazilevič function of type β when $\rho = 0$. We notice that if $\beta = 1$ then $B(1, 0) = K(0)$.

By Bazilevič *et al.* it is easily verified that if $f(z) \in B(\beta, \rho)$ with $0 < \beta$ and $0 \leqq \rho < 1$, then $f(z) \in S$.

Let $A(\alpha, \beta)$ be the subclass of A consisting of functions satisfying

$$\operatorname{Re} \left\{(1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta$$

for all $z \in E$, for some real α, β.

The class $A(\alpha, 0)$ when $\beta = 0$ was introduced by Mocanu[5] and was studied by Miller *et al.*[4] and Sakaguchi and Fukui[9] among others.

2. Main Theorem

We begin with the statement and the proof of our main result.

Theorem—If $f(z) \in A(\alpha, \beta)$ with $\alpha > 0$ and $|\beta/\alpha| \leqq 1/2$, then $f(z)$ is univalent in E. Moreover, if $0 \leqq -\beta/\alpha \leqq 1/2$, then $f(z) \in B(1/\alpha, 2^{2\beta/\alpha})$.

Proof : Let us put

$$H(z) = \frac{1}{1-\beta}\left\{(1-\alpha) \frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - \beta\right\}$$

$$= \frac{zg'(z)}{g(z)}$$

for $f(z)$ in $A(\alpha, \beta)$. By the definition of $A(\alpha, \beta)$, we have $H(0) = 1$ and $\operatorname{Re} H(z) > 0$ in E. It follows from this that $g(z) \in S^*$. A simple calculation gives

$$f'(z)^{\alpha/(1-\beta)} = \frac{g(z)}{f(z)^{(1-\alpha)/(1-\beta)} z^{(\alpha-\beta)/(1-\beta)}}$$

and therefore we have

$$\frac{z f'(z)}{f(z)^{1-1/\alpha} g(z)^{1/\alpha}} = \left\{ \frac{g(z)}{z} \right\}^{-\beta/\alpha}.$$

Using the result due to Komatu[3] that

$$\frac{g(z)}{z} \prec \frac{1}{(1-z)^2} \quad (z \in E)$$

for $g(z)$ in S^*, where the symbol $\prec$ denotes subordination, we have

$$\left\{ \frac{g(z)}{z} \right\}^{-\beta/\alpha} \prec \left\{ \frac{1}{1-z} \right\}^{-2\beta/\alpha} \qquad (z \in E).$$

This shows that

$$\text{Re} \frac{z f'(z)}{f(z)^{1-1/\alpha} \; g(z)^{1/\alpha}} > 0 \; (z \in E)$$

for $|\beta/\alpha| \leqq 1/2$. Moreover, if $0 \leq -\beta/\alpha \leqq 1/2$, then we have

$$\text{Re} \frac{zf'(z)}{f(z)^{1-1/\alpha} g(z)^{1/\alpha}} > 2^{2\beta/\alpha} \qquad (z \in E)$$

which proves that $f(z)$ is a Bazilevič function of type $1/\alpha$ and order $2^{3\beta/\alpha}$. This completes the proof of Theorem.

Letting $\beta = 0$ in the Theorem, we have

Corollary 1—If $f(z) \in A(\alpha, 0)$ with $\alpha > 0$, then $f(z) \in B(1/\alpha, 1)$. Further letting $\alpha = 1$ in the Theorem yields

Carollary 2—If $f(z)$ belongs to A and satisfies

$$1 + \text{Re} \frac{z f''(z)}{f'(z)} > \beta \qquad (z \in E)$$

for $-1/2 \leqq \beta < 0$, then $f(z) \in K(2^{2\beta})$ and therefore $f(z)$ is close-to-convex of order $2^{2\beta}$.

Ozaki[7] showed that if $f(z) \in A$ satisfies

$$1 + \text{Re} \frac{z f''(z)}{f'(z)} > -\tfrac{1}{2} \qquad (z \in E), \quad \ldots \quad (2.1)$$

then $f(z)$ is univalent in E and convex in some direction. By virture of Corollary 2, we find that $f(z) \in A$ satisfies the condition (2.1), then $f(z)$ is close-to-convex of order 1/2.

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References

1. E. Bazilevič, *Mat. Sb.* **37** (1955), 471–76 (Russian).
2. W. Kaplan, *Mich. Math. J.* **1** (1952), 169–85.
3. Y. Komatu, *Kodai Math. Sem. Rep.* **13** (1961), 123–26.
4. S. S. Miller, P. T. Mocanu, and M. O. Reade, *Proc. Am. Math. Soc.* **37** (1973), 553–54.
5. P. T. Mocanu, *Mathematica Cluj* **11** (1969), 127–33.
6. S. Ozaki, *Sci. Rep. Tokyo Bunrika Daigaku* A2 (1935), 167–88.
7. S. Ozaki, *Sci. Rep. Tokyo Bunrika Daigaku* **4** (1941), 45–86.
8. J. A. Pfaltzgraff, M. O. Reade, and T. Umezawa, *Ann. Fac. Sci. Kinshasa, Zaire; Section Math. Phys.* **2** (1976), 94–101.
9. K. Sakaguchi, and S. Fukui, *Bull. Nara Univ. of Ed.* **28** (1979), 5–12.
10. D. K. Thomas, *Trans. Am. Math. Soc.* **132** (1968), 353–61.
11. T. Umezawa, *J. Math. Soc. Japan* **4** (1952), 194–202.

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ON EXTENSIONS OF FUGLEDE-PUTNAM THEOREM

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An operator A on a separable complex Hilbert space is called k-quasihyponormal if $A^{*k}(A^*A-AA^*)A^k \geqslant 0$. If $A^{*k}A^k \geqslant (A^*A)^k$, then A is defined to be of class $(M;k)$. The classical Fuglede-Putnam theorem states that if A, B are normal operators and $AX = XB$ for some operator X, then $A^*X = XB^*$, We prove that if $\{A, C\}$, $\{B, D\}$are doubly commuting pairs and $AXD = CXB$ for some operator $X \in C_2$, then $A^*XD^* = C^*XB^*$ under any one of the following conditions:

(i) A, C, B^*, D^* are k-quasihyponormal operators and B and C are invertible

(ii) A, D^* are in $(M; k)$; and B^* and C are invertible with ther inverses being in $(M; k)$.

We also show that if A is 1-quasihyponormal contraction and $I^*—AA^* \in C_1$ then for any operator X, $AX - XA \in C_2$ implies $A^*X - XA \in C_2$.

By an operator we mean a bounded linear operator on a separable complex Hilbert space H. Let $B(H)$, C_1 and C_2 denote respectively the Banach algebra of all operators, the trace class and the Hilbert-Schmidt class of operators on H and let $\| \|_2$ denote the Hilbert-Schmidt norm on C_2. The class C_2 is a Hilbert space with the inner product $\langle X, Y \rangle = \mathrm{Tr}\,(Y^*X)$.

An operator A is called hyponormal if $A^*A - AA^* \geqslant 0$; k-quasihyponormal if $A^{*k}(A^*A - AA^*)A^k \geqslant 0$, k being a positive integer. It is clear that the class $Q(k)$ of all k-quasihyponormal operators contains all hyponormal operators properly and forms a strictly increasing sequence in k; and every invertible operator in $Q(k)$ is hyponormal. The class $Q(k)$ has been studied earlier[2,5,6]. Inspite of the fact that $Q(k)$ contains a large class of operators other than hyponormal operators, its behaviour in many respect is similar to that of hyponormal operators.

If $A^{*k}A^k \geqslant (A^*A)^k$, then A is defined to be of class $(M; k)$. The class $(M; k)$ was introduced and studied by Patel[9]. Clearly the class $Q(1)$ coincides with the class $(M; 2)$ and it is noted[9] that for $k > 2$, every hyponormal operator need not be in the class $(M; k)$ and vice-versa.

The classical Fuglede-Putnam theorem states that if A and B are normal operators and $AX = XB$ for some $X \in B(H)$, then $A^*X = XB^*$ (Halmos[7]). This result has been extended to various classes of non-normal operators including hyponormal and k-quasi-hyponormal operators[1,3,4,6,10]. Generalizing Fuglede-Putnam theorem, Weiss[10] proved that if $\{A, C\}$ and $\{B, D\}$ are commuting pairs of normal operators then for any $X \in B(H)$,

$$\|AXD + CXB\|_2 = \|A^*XD^* + C^* XB^*\|_2.$$

In particular,

(i) $A^*XD^* - C^*XB^* \in C_2$ whenever $AXD - CXB \in C_2$

and (ii) $A^*XD^* = C^*XB^*$ whenever $AXD = CXB$.

Furuta[4] extended the above result of Weiss to hyponormal operators with the additional hypothesis that $X \in C_2$ and proved the following.

Theorem A—Suppose A, C, B^* and D^* are hyponormal operators such that $\{A, C^*\}$ and $\{B, D^*\}$ are commuting pairs. Then for any $X \in C_2$,

$$\|AXD+CXB\|_2 \geq \|A^*XD^*+C^*XB^*\|_2.$$

In particular, he obtained the generalized Fuglede-Putnam theorem which states that if $AXD = CXB$ and $X \in C_2$, then $A^*XD^* = C^*XB^*$. When $C = D = I$, this theorem includes a result due to Berberian[1].

Let us say that operators A and B doubly commute if A commutes with B and B^* both. In this paper, we show that in the generailzed Fuglede-Putnam theorem proved by Furuta, the conditions on A, C, B^* and D^* can further be weakened by requiring that B^* and C are invertible and $\{A, C\}$ and $\{B, D\}$ are doubly commuting pairs. Finally, we show that if $A \in Q(1)$ is a contraction and $I - AA^* \in C_1$ then for any $X \in B(H)$, $AX - XA \in C_2$ implies $A^*X - XB^* \in C_2$.

For given operators A, B in $B(H)$, let $\mathcal{T}$ be defined on C_2 by $\mathcal{T} X = AXB$. Then $\mathcal{T}$ is a bounded linear operator on the Hilbart space C_2 and $\mathcal{T}^* X = A^* XB^*$ (Berberian[1]). The following lemma is due to Furuta[3].

Lemma 1—If A and B^* are in $Q(k)$, then $\mathcal{T}$ is also in $Q(k)$.

We first prove a similar result for operators in $(M; k)$.

Lemma 2—If A and B^* are in $(M; k)$, then $\mathcal{T}$ is in $(M; k)$.

PROOF : $(\mathcal{T}^{*k} \mathcal{T}^k - (\mathcal{T}^* \mathcal{T})^k)X$

$$\begin{aligned} &= A^{*k} A^k XB^k B^{*k} - (A^*A)^k X(BB^*)^k \\ &= A^{*k} A^k XB^k B^{*k} - (A^*A)^k X(B^kB^{*k}) + (A^*A)^k XB^k B^{*k} \\ &\qquad - (A^*A)^k X(BB^*)^k \end{aligned}$$

(equation continued on p. 57)

$$= [A^{*k}A^k - (A^* A)^k] XB^k B^{*k} + (A^* A)^k X [B^k B^{*k} - (BB^*)^k].$$

Since the left and right multiplications on C_2 by positive operators are again positive and A, B^* are in $(M; k)$, it follows that $\mathcal{T}^{*k} \mathcal{T}^k - (\mathcal{T}^* \mathcal{T})^k$ is the sum of two positive operators and thus $\mathcal{T} \in (M; k)$.

Theorem 1—Suppose (i) A, C, B^* and D^* are in $Q(k)$; (ii) B and C are invertible and (iii) $\{A, C\}$ and $\{B, D\}$ are doubly commuting pairs. If $AXD = CXB$ and $X \in C_2$, then $A^* XD^* = C^* XB^*$.

Proof : Define $\mathcal{T}$ on C_2 by $\mathcal{T} Y = C^{-1} AYDB^{-1}$. Since B^* and C are invertible operarors in $Q(k)$, they are hyponormal operators and hence B^{*-1} and C^{-1} are in $Q(k)$. It can be easily seen that the product of doubly commuting operators in $Q(k)$ are again in $Q(k)$. Therefore, $C^{-1} A$ and $(DB^{-1})^*$ are in $Q(k)$; and by Lemma 1, $\mathcal{T} \in Q(k)$. Now if $AXD = CXB$ for $X \in C_2$, then $\mathcal{T} X = X$. Since non-zero eigenvalues of an operator in $Q(k)$ are reducing[2], we have $\mathcal{T}^* X = X$ and follows that $A^*XD^* = C^*XB^*$.

Theorem 2—Let A, B, C and D be operators such that (i) A and D^* are in $(M; k)$; (ii) B^* and C are invertible with their inverses being in $(M; k)$; and (iii) $\{A, C\}$ and $\{B, D\}$ are doubly commuting pairs. If $AXD = CXB$ and $X \in C_2$, then $A^*XD^* = C^*XB^*$.

Proof : Since the product of doubly commuting operators of class $(M; k)$ is again in $(M; k)$, we have $C^{-1} A$ and $(D B^{-1})^*$ and the operator $\mathcal{T}$ defined on C_2 by $\mathcal{T} Y = C^{-1} AYDB^{-1}$ all in $(M; k)$. Moreover by the hypothesis, $\mathcal{T} X = X$. Invoking (Patel[9], Theorem 4), we get $\mathcal{T}^* X = X$, that is, $A^*XD^* = C^*XB^*$.

The following result extends Theorem 3 of Kittaneh[8] to the operators of class $Q(1)$.

Theorem 3—If A is a contraction operator in $Q(1)$ and $I - AA^* \in C_1$, then $A^*X - XA^* \in C_2$ whenever $AX - XA \in C_2$.

Proof : Let $A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}$ be the martix representation of A with respect to the decomposition $H = \overline{R(A)} \oplus N(A^*)$, where A_1 is hyponormal[2]. Since $I - AA^* \in C_1$, we have $\dim N(A^*) < \infty$. Thus A is of the form $S + F$, where

$S = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$ is hyponormal and $F = \begin{bmatrix} 0 & A_2 \\ 0 & 0 \end{bmatrix}$ is of finite rank. Therefore $I - SS^* = I - AA^* + FF^* \in C_1$. Now the rest of the argument for the desired assertion follows exactly on the same lines as that of Theorem 3 of Kittaneh[8].

References

1. S. K. Berberian. *Proc. Am. Math. Soc.* 71 (1978), 113-14.
2. S. L. Campbell, and B. C. Gupta, *Math. Joponica* 23 (1978), 185-89.

3. T. Furuta, *Proc. Am. Math. Sac.* 77 (1979), 324–28.
4. T. Furuta, *Bull. Austr. Math. Soc.* 25 (1982,) 177–85.
5. B. C. Gupta, and P. B. Ramanujan, *Bull. Austr. Math Soc.* **24** (1981), 61–67.
6. B. C. Gupta, *Indian J. pure appl. Math* **14** (1983), 1343–47.
7. P. R. Halmos, *A Hilbert Space Problem Book*, Springer-Verlag, New York, 1974.
8. F. Kittaneh, *Proc. Am. Math. Soc.* **88** (1983), 293–98.
9. S. M. Patel, *Math. Nachr.* 73 (1975), 147–50.
10. G. Weiss, *J. Operator Theory* 5 (1981), 3-16.

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GENERAL GENERATING RELATIONS

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Following the work of Srivastava and Buschman[7], a number of generating functions have been derived here for the set of polynomials $\left\{ f_n^{c} (x, y, r, m) \right\}$ introduced into analysis by Singhal and Savita Kumari[4].

1. INTRODUCTION

Earlier in an attempt to present a unified treatment of the various polynomial systems introduced into analysis from time to time, and to provide an extension of a theorem due to Mittal[3], Srivastava and Buschman[7], gave the following theorem :

Corresponding to the power series

$$F(u) = \sum_{n=0}^{\infty} \gamma_n u^n, \; \gamma_0 \neq 0 \qquad \ldots(1.1)$$

let

$$S_{n,q}^{(\alpha,\beta)} (\lambda ; x) = \sum_{k=0}^{[n/q]} \frac{(-n)_{qk} (1 + \alpha + (\beta + 1)n)_{\lambda k}}{(1 + \alpha + \beta n) (\lambda + q)_k} \gamma_k x^k \qquad \ldots(1.2)$$

and

$$\theta (n, q; \alpha, \beta, \gamma, \lambda; u)$$

$$= \sum_{k=0}^{\infty} \frac{\gamma}{\gamma + (\beta + 1) qk} \binom{\alpha - \gamma + \lambda k}{n} \binom{n + qk + \gamma/(\beta + 1)}{n}^{-1} \gamma_k u^k$$

$$\ldots(1.3)$$

where α, β, γ and λ are arbitrary complex numbers, q is a positive integer, and $n = 0, 1, 2, \ldots$

then

$$\sum_{n=0}^{\infty} \frac{\gamma}{\gamma + (\beta + 1) n} \binom{\alpha + (\beta + 1) n}{n} S_{n,q}^{(\alpha,\beta)} (\lambda; x) t^n$$

$$= (1 + w)^{\alpha} \phi (x (- w)^q (1 + w)^{\lambda}, - w/(1 + w)) \qquad \ldots(1.4)$$

where, for convenience,

$$\phi\,(u, v) = \sum_{n=0}^{\infty} \theta\,(n, q; \alpha, \beta, \gamma, \lambda; u)\, v^n \qquad \ldots(1.5)$$

and

$$w = t\,(1 + w)^{\beta+1},\ w\,(0) = 0. \qquad \ldots(1.6)$$

The polynomial set $\left\{S_{n,q}^{(\alpha,\beta)}\,(\lambda; x)\right\}$ considered by Srivastava and Buschman[7], though quite general, does not include the set of polynomials $\left\{f_n^c\,(x, y, r, m)\right\}$ introduced by the authors[4,5]. As such the above theorem is not applicacle to $\left\{f_n^c\,(x,y,r,m)\right\}$. It would, therefore, be of interest to obtain the analogue of the above mentioned theorem (which we give in section 2), involving the polynomials $f_n^c\,(x, y, r, m)$ abbreviated as $f_n^c\,(x)$ here-in-after wherever there is no ambiquity regarding the other parameters.

2. General Generating Relation

Theorem 1—Corresponding to the power-series $F\,(u)$ given by (1.1)

let

$$f_n^c\,(x) = \sum_{k=0}^{[n/m]} \frac{(rn - rmk + c)_k}{k!}\,(-\,y)^k\,\gamma_{n-mk}\,x^{n-mk} \qquad \ldots(2.1)$$

and

$$S_n\,(m, c, \sigma, r, u)$$

$$= \sum_{k=0}^{\infty} \frac{\gamma}{\gamma - \sigma k}\binom{-\,c - \gamma - rk}{n}\binom{n + k/m - \gamma/(\sigma m)}{n}^{-1}\gamma_k\,u^k \quad \ldots(2.2)$$

where c, γ, σ are arbitrary complex numbers, r is any integer, m is a positive integer and $n = 0, 1, 2, \ldots$

Then

$$\sum_{k=0}^{\infty} \frac{\gamma}{\gamma - \sigma^n}\,f_n^{c+\sigma n}\,(x)t^n$$

$$= (1 + y\,\xi^m)^{-c}\;\Phi\left(\frac{x\xi}{(1 + y\,\xi^m)^r},\ \frac{-\,y\,\xi^m}{(1 + y\xi^m)}\right) \qquad \ldots(2.3)$$

where

$$\xi = t\,(1 + y\,\xi^m)^{-\sigma},\ \xi\,(0) = 0 \qquad \ldots(2.4)$$

and, for convenience

$$\Phi(u, v) = \sum_{n=0}^{\infty} S_n(m, c, \sigma, r, u)\, v^n. \qquad \ldots(2.5)$$

In order to prove this theorem, we first observe that $f_n^c(x)$ may be expressed in the form

$$f_n^c(x) = \sum_{k=0}^{[n/m]} \binom{-c - rn + rmk}{k} y^k \gamma_{n-mk}\, x^{n-mk} \qquad \ldots(2.6)$$

so that

$$\sum_{n=0}^{\infty} \frac{\gamma}{\gamma - \sigma n} f_n^{c+\sigma n}(x)\, t^n$$

$$= \sum_{n=0}^{\infty} \frac{\gamma}{\gamma - \sigma n} (xt)^n \gamma_n \sum_{k=0}^{\infty} \frac{\gamma - \sigma n}{\gamma - \sigma n - \sigma mk}$$

$$\binom{-c - \sigma n - rn - \sigma mk}{k} (yt^m)^k. \qquad \ldots(2.7)$$

The inner series may be transformed by making use of Gould's identity (Gould[1], p. 196)

$$\sum_{k=0}^{\infty} \frac{a}{a + bk} \binom{c + bk}{k} t^k$$

$$= (1 + w)^c \sum_{k=0}^{\infty} (-1)^k \binom{c - a}{k} \binom{k + a/b}{k}^{-1} \left(\frac{w}{1 + w}\right)^k$$

$$\ldots(2.8)$$

where $w = t(w + 1)^b$.

As a result, the right hand member of (2.7) becomes

$$(1 + y\,\xi^m)^{-c} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\gamma}{\gamma - \sigma n} \binom{-c - \gamma - rn}{k} \binom{k + n/m - \gamma/(\sigma m)}{k}^{-1}$$

$$\left(\frac{x\xi}{(1 + y\xi m)^r}\right)^n \left(-\frac{y\xi^m}{1 + y\xi^m}\right)^k \gamma_n \qquad \ldots(2.9)$$

where from the theorem follows, at once.

On the other hand if we express the binomial coefficient occurring in (2.6) in the form

$$\binom{-c-rn+rmk}{k} = (-1)^k \binom{c+rn-rmk+k-1}{k} \qquad ...(2.10)$$

and follow an analysis similar to Theorem 1 we shall get another theorem in the form:

Theorem 2—In terms of power series $F(u)$ by (1.1) let

$$T_n(m, c, \sigma, r, u)$$

$$= \sum_{k=0}^{\infty} \frac{\gamma}{\gamma + (\sigma + 1/m)^k} \binom{c-\gamma-1+rk/m}{n} \binom{n+k/m+\gamma/(m\sigma+1)}{n}^{-1}$$

$$\times \gamma_k u^k \qquad ...(2.11)$$

where, as before, c, γ, σ are arbitrary complex numbers, r is any integer, m is a positive integer and $n = 0, 1, 2, \ldots$

Then

$$\sum_{n=0}^{\infty} \frac{\gamma}{\gamma + (\sigma + 1/m)n} f_n^{c+\sigma n}(x)\, t^n$$

$$= (1+w)^{c-1} K(xt(1+w)^{r+\sigma}, -w/(1+w)) \qquad ...(2.12)$$

where

$$w = -yt^m (w+1)^{\sigma m+1}, \; w(0) = 0 \qquad ...(2.13)$$

and

$$K(u, v) = \sum_{n=0}^{\infty} T_n(m, c, \sigma, r, u)\, v^n. \qquad ...(2.14)$$

3. Particular Cases

When $\sigma \to 0$, it is easy to see that $S_n \to 0$ for $n = 1, 2, \ldots$ and $S_0 \to F(u)$, therefore the generating relation (2.3) — (2.5) would reduce to the generating relation [Singhal and Savita Kumari[4], eqns. (1.4), (1.5)]

$$(1+yt^m)^{-c}\, G\left[\frac{xt}{(1+yt^m)^r}\right] = \sum_{n=0}^{\infty} f_n^c(x) t^n \qquad ... (3.1)$$

where

$$G[z] = \sum_{n=0}^{\infty} \gamma_n z^n, \; \gamma_0 \neq 0. \qquad ...(3.2)$$

Next, we observe that $S_n \to (m\sigma)^n F(u)$ and

$$\Phi(u, v) \to \frac{F(u)}{1 - m\sigma v}, \text{ when } \gamma \to \infty.$$

Consequently, the particular case $\gamma \to \infty$ of (2.3) -- (2.5) would yield the generating relation [Singhal and Savita Kumari[4], eqns. (2.1), (2.2)]

$$\sum_{n=0}^{\infty} f_n^{c+\sigma n}(x)\, t^n = \frac{(1 + yw^m)^{1-c}}{1 + y(1 + \sigma m)w^m}\, G\left[\frac{xw}{(1 + yw^m)^r}\right] \quad \text{...(3.3)}$$

$$w = t(1 + yw^m)^{-\sigma},\ w(0) = 0 \quad \text{...(3.4)}$$

and $G[z]$ is given by (3.2).

Further, when $\gamma = c - 1$, $r = 1/m$, the generating relation (2.12) — (2.14) would reduce to

$$\sum_{n=0}^{\infty} \frac{c - 1}{c - 1 + (\sigma + 1/m)n} f_n^{c+\sigma n}(x, y, 1/m, m)\, t^n$$

$$= u^{c-1} H[xt\, u^{\sigma+1/m}] \quad \text{...(3.5)}$$

where

$$u = 1 - yt^m u^{\sigma m+1},\ u(0) = 1 \quad \text{...(3.6)}$$

and

$$H(z) = \sum_{k=0}^{\infty} \frac{c - 1}{c - 1 + (\sigma + 1/m)k} \gamma_k z^k. \quad \text{...(3.7)}$$

The above generating relation may also be expressed in the alternative from :

$$\sum_{n=0}^{\infty} \frac{c - 1}{c - 1 + (\sigma + 1/m)n} f_n^{c+\sigma n}(x, y, 1/m, m)t^n$$

$$= (1 + y\eta^m)^{c-1} H[(-1)^{1/m} x\eta] \quad \text{...(3.8)}$$

where

$$\eta = (-1)^{1/m} t(1 + y\eta^m)^{\sigma+1/m} \quad \text{...(3.9)}$$

and $H(z)$ is given by (3.7).

The substitution $\sigma = -2/m$ transforms (3.5) – (3.7) to

$$\sum_{n=0}^{\infty} \frac{c - 1}{c - 1 - n/m} f_n^{c-2n/m}(x, y, 1/m, m)\, t^n$$

(*equation continued on p. 64*)

$$= \left(\frac{1+\sqrt{1-4yt^m}}{2}\right)^{c-1} \Phi\left[xt\left(\frac{1+\sqrt{1-4yt^m}}{2}\right)\right]^{-1/m} \qquad \ldots(3.10)$$

where

$$\Phi(z) = \sum_{k=0}^{\infty} \frac{c-1}{c-1-k/m} \gamma_k z^k \qquad \ldots(3.11)$$

whereas for $\sigma = -1/m$ it evidently reduces to [Singhal and Savita Kumari[4], eqn. (2.8)] with $r = 1/m$, that is

$$\sum_{n=0}^{\infty} f_n^{c-n/m}(x, y, 1/m, m) t^n = (1 - yt^m)^{c-1}\, G[xt]. \qquad \ldots(3.12)$$

On the other hand on putting $\sigma = 0$, we shall get

$$\sum_{n=0}^{\infty} \frac{c-1}{c-1+n/m} f_n^c(x, y, 1/m, m)\, t^n$$

$$= (1 + yt^m)^{1-c}\, \theta\left(\frac{xt}{(1+yt^m)^{1/m}}\right) \qquad \ldots(3.13)$$

where

$$\theta(z) = \sum_{k=0}^{\infty} \frac{c-1}{c-1+k/m} \gamma_k z^k \qquad \ldots(3.14)$$

and on putting $\sigma = 1/m$, (3.5) — (3.7) would reduce to

$$\sum_{n=0}^{\infty} \frac{c-1}{c-1+2n/m} f_n^{c+n/m}(x, y, 1/m, m) t^n$$

$$= \left(\frac{1+\sqrt{1+4yt^m}}{2}\right)^{1-c} \Phi\left[xt\left(\frac{1+\sqrt{1+4yt^m}}{2}\right)^{-2/m}\right] \qquad \ldots(3.15)$$

where

$$\phi(z) = \sum_{k=0}^{\infty} \frac{c-1}{c-1+2k/m} \gamma_k z^k. \qquad \ldots(3.16)$$

The relation (3.5) — (3.7) may also be stated in the form of following theorem :

Theorem 3—Let $\{p_n(x)\}$ be the polynomials possessing the generating relation,

$$\sum_{n=0}^{\infty} p_n(x)t^n = \frac{(1+yw^m)^{1-c}}{1+y(1+\sigma m)w^m} G\left[\frac{xw}{(1+yw^m)^{1/m}}\right] \qquad ...(3.17)$$

where w is given by (3.4) and $G[z]$ is given by (3.2), then

$$\sum_{n=0}^{\infty} \frac{c-1}{c-1+(\sigma+1/m)n} p_n(x)t^n = u^{c-1} H[xt\, u^{\sigma+1/m}] \qquad ...(3.18)$$

where u and H are given by (3.6) and (3.7) respectively.

When $\sigma = -2, c = 1-a, y = 1$ and $m = 1$, Theorem 3 would particularize to the known Theorem 1 given in McBride[2] (p. 85) whereas if we put $\sigma = 1, y = -1$, $m = 1$ and replace c by $1 + a$, our Theorem 3 would correspond to its particular case given in McBride[2] (Theorem 2, p. 85)

Another worth mentioning particular case of our Theorem 3 is the known result

$$\sum_{n=0}^{\infty} \frac{\alpha}{\alpha+(\beta+1)n} L_n^{(\alpha+\beta n)}(x)t^n$$

$$= (1+\xi)^{\alpha} {}_1F_1\left[\begin{matrix}\alpha/(\beta+1); \\ 1+\alpha/(\beta+1);\end{matrix} \; -x\xi\right] \qquad ...(3.19)$$

where

$$\xi = t(1+\xi)^{\beta+1}, \; \xi(0) = 0$$

derived earlier by Srivastava[6] (eqn. 2.4).

References

1. H. W. Gould, *Duke Math. J.* 28 (1961), 193-202.
2. Elna B. McBride, *Obtaining Generating Functions*. Springer-Verlag, New York, Heidelberg and Berlin, 1971.
3. H. B. Mittal, *Trans. Am. Math. Soc.* 168 (1972), 73-84.
4. J. P. Singhal, and Savita Kumari, *Jnaanbha* 9/10 (1980), 171-78.
5. J. P. Singhal, and Savita Kumari, *Bull. Inst. Math. Acad. Sinica* 10 (1982), 171-75.
6. H. M. Srivastava, *Bull. Inst. Math. Acad. Sinica* 8 (1980), 571-79.
7. H. M. Srivastava, and R. G. Buschman, *Trans. Am. Math. Soc.* 205 (1975) 360-70.

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AN IMPLICIT NAVIER-STOKES SOLVER FOR TWO-DIMENSIONAL TRANSONIC SHOCK WAVE-BOUNDARY LAYER INTERACTION

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A code has been developed using an implicit numerical scheme for solving the two-dimensional unsteady Navier-Stokes equations for the transonic shock wave-boundary layer interaction problem. The method is an adaptation of the implicit Mac Cormack scheme for viscous compressible flows. The technique is illustrated by applying it to study the flow past an 18% symmetric circular arc airfoil. A coordinate transformation is made use of to facilitate direct application of the method. Numerical results obtained through the software, for surface pressure distribution and skin friction exhibit reasonable agreement with those got earlier by Deiwert using an explicit integral scheme.

1. INTRODUCTION

The problem of tronsonic shock wave-boundary layer interaction is of considerable importance in the context of aerodynamic design and has been currently engaging the attention of researchers[1,2]. The present paper is an attempt to numerically simulate the flow field for this problem with the aid of an implicit Navier-Stokes solver. The scheme is analogous to MacCormack's implicit numerical method for solving unsteady two-dimensional Navier Stokes equations[3–13].

The details of the flow field under investigation are depicted in Fig. 1. The freestream flow is subsonic and a supersonic flow region is generated between the

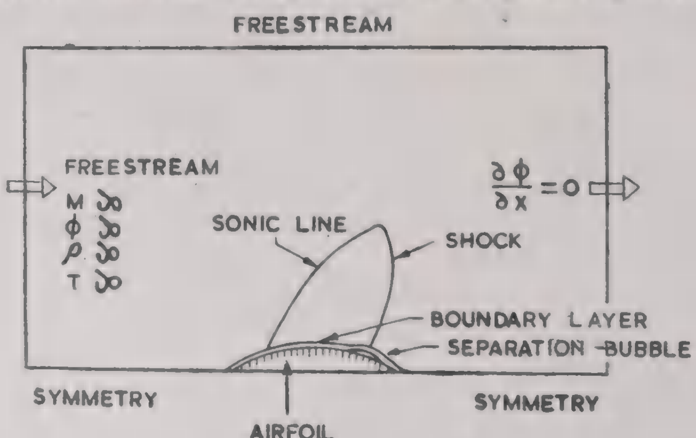

FIG. 1. Flow past circular arc airfoil.

leading and trailing edges of the airfoil. This supersonic flow becomes subsonic by passing through a standing shock. If the shock is sufficiently strong, boundary layer separation can occur.

2. Governing Equations and Transformation

The two-dimensional unsteady Navier-Stokes equations may be written in Cartesian coordinates in conservation form as

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \qquad \ldots(1)$$

where

$$U = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{Bmatrix}, \quad F = \begin{Bmatrix} \rho u \\ \rho u^2 + \sigma_x \\ \rho uv + \tau_{xy} \\ (e + \sigma_x) u + \tau_{yx} v - k \dfrac{\partial T}{\partial x} \end{Bmatrix}$$

$$G = \begin{Bmatrix} \rho v \\ \rho uv + \tau_{yx} \\ \rho v^2 + \sigma_y \\ (e + \sigma_y) v + \tau_{xy} u - k \dfrac{\partial T}{\partial y} \end{Bmatrix} \qquad \ldots(2)$$

and where

$$\left.\begin{aligned} \sigma_x &= p - \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - 2\mu \frac{\partial u}{\partial x} \\ \sigma_y &= p - \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - 2\mu \frac{\partial v}{\partial x} \\ \tau_{xy} &= \tau_{yx} = -\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \end{aligned}\right\} \qquad \ldots(3)$$

in terms of density ρ, x- and y-velocity components u and v, viscosity coefficients λ and μ, total energy per unit volume e, coefficient of heat conductivity k and temperature T. Finally, the pressure p is related to the specific internal energy e_I and ρ by an equation of state $p(e_I, \rho)$, where

$$e_I = \frac{e}{\rho} - \frac{u^2 + v^2}{2}.$$

The transformation $\xi = x$, $\eta = y - R(x)$, where $y = R(x)$ represents the equation of the body surface, is made use of to obtain rectangular computational plane. Equation. (1) then becomes

$$\frac{\partial U}{\partial t} + \frac{\partial F'}{\partial \xi} + \frac{\partial G'}{\partial \eta} = 0 \qquad \ldots(4)$$

where

$$F' = F, G' = G - \frac{dR}{dx} F. \qquad \ldots(5)$$

For an 18% symmetric circular arc airfoil of chord length unity, for instance, the equation of the body surface, referred to rectangular coordinates with the x-axis along the chord and origin at one end of the chord, is given by

$$x^2 + y^2 - x + 2.68778y = 0 \qquad \ldots(6)$$

and, consequently, the transformed cooıdinate $\eta = \eta\,(x, y)$is obtained as

$$\eta = y - \tfrac{1}{2}\{- 2.68778 + [7.22416 - 4x - 4x^2]^{1/2}\}. \qquad \ldots(7)$$

3. Implicit Finite-Difference Schme

Following[3,7], an implicit predictor-corrector scheme may be defined for the numerical integration of the transformed equations as follows :

Predictor

$$\Delta U^n_{ij} = - \Delta t\left(\frac{D_+}{\Delta\xi} F'^n_{ij} + \frac{D_+}{\Delta\eta} G'^n_{ij}\right).$$

$$\left(I - \Delta t \frac{D_+}{\Delta\xi} \mid A \mid .\right)\left(I - \Delta t \frac{D_+}{\Delta\eta} \mid B \mid .\right) \delta\, \overline{U^{n+1}_{ij}} = \Delta U^n_{ij}$$

$$\overline{U^{n+1}_{ij}} = U^n_{ij} + \delta\, \overline{U^{n+1}_{ij}}.$$

Corrector

$$\Delta \overline{U^{n+1}_{ij}} = - \Delta t\left(\frac{D_-}{\Delta\xi} \overline{F'^{n+1}_{ij}} + \frac{D_-}{\Delta\eta} G'^{n+1}_{ij}\right)$$

$$\left(I + \Delta t \frac{D_-}{\Delta\xi} \mid A \mid .\right)\left(I + \Delta t \frac{D_-}{\Delta\eta} \mid B \mid .\right) \delta U^{n+1}_{ij} = \Delta \overline{U^{n+1}_{ij}}$$

$$U^{n+1}_{ij} = \tfrac{1}{2}\left[U^n_{ij} + \overline{U^{n+1}_{ij}} + \delta\, \overline{U^{n+1}_{ij}}\right]. \qquad \ldots(8)$$

The matrices $\mid A \mid$ and $\mid B \mid$ have positive eigenvalues and are related to the Jacobians $A\left(= \frac{\partial F'}{\partial U}\right)$ and $B\left(= \frac{\partial G'}{\partial U}\right)$. Let S_ξ and S_η and their inverses denote the matrices that diagonalise A and B with $\mu = \lambda = k = 0$, i.e. with the viscous terms neglected. Then, with perfect gas relations, A and B may be arranged as

$$A = S_\xi^{-1}\, \Lambda_A\, S_\xi$$

and

$$B = S_\eta^{-1}\, \Lambda_B\, S_\eta \qquad \ldots(9)$$

where

$$S_\xi = \begin{bmatrix} 1-\dfrac{\alpha\beta}{C^2} & \dfrac{u\beta}{C^2} & \dfrac{v\beta}{C^2} & -\dfrac{\beta}{C^2} \\ -uc+\alpha\beta & c-u\beta & -v\beta & \beta \\ -\dfrac{v}{\rho} & 0 & \dfrac{1}{\rho} & 0 \\ uc+\alpha\beta & -c-u\beta & -v\beta & \beta \end{bmatrix}$$

$$S_\eta = \begin{bmatrix} 1-\dfrac{\alpha\beta}{c^2} & \dfrac{u\beta}{c^2} & \dfrac{v\beta}{c^2} & -\dfrac{\beta}{c^2} \\ \dfrac{\eta_x v-\eta_y u}{\rho\alpha_2} & \dfrac{\eta_y}{\rho\alpha_2} & -\dfrac{\eta_x}{\rho\alpha_2} & 0 \\ \beta_1\left(\alpha\beta - c\dfrac{\alpha_1}{\alpha_2}\right) & \beta_1\left(\dfrac{\eta_x c}{\alpha_2}-u\beta\right) & \beta_1\left(\dfrac{\eta_y c}{\alpha_2}-v\beta\right) & \beta_1\beta \\ \beta_1\left(\alpha\beta + c\dfrac{\alpha_1}{\alpha_2}\right) & -\beta_1\left(\dfrac{\eta_x c}{\alpha_2}+u\beta\right) & -\beta_1\left(\dfrac{\eta_y c}{\alpha_2}+v\beta\right) & \beta_1\beta \end{bmatrix} \quad \ldots(10)$$

$$\alpha_1 = \eta_x u + \eta_y v, \quad \alpha_2 = (\eta_x^2 + \eta_y^2)^{1/2}$$

$$\alpha = \tfrac{1}{2}(u^2+v^2),\ \beta = \gamma - 1 \text{ and } \beta_1 = \frac{1}{\sqrt{2}\rho c}.$$

Λ_A and Λ_B are diagonal matrices with diagonal elements $(u, u+c, u, u-c)$ and $(\alpha_1, \alpha_1, \alpha_1 + c\alpha_2, \alpha_1 - c\alpha_2)$, respectively

The matrices $|A|$ and $|B|$ are now defined by

$$|A| = S_\xi^{-1} D_A S_\xi$$

and

$$|B| = S_\eta^{-1} D_B S_\eta \quad \ldots(11)$$

where D_A and D_B are diagonal matrices with diagonal elements λ_{Ai}, $i = 1, \ldots, 4$ and λ_{Bi}, $i = 1, \ldots, 4$, λ_{Ai} and λ_{Bi} being formed as

$$\lambda_{A1} = \mathrm{Max}\left\{ |u| + \frac{2\nu}{\rho\Delta\xi} - \frac{1}{2}\frac{\Delta\xi}{\Delta t}, 0.0 \right\} \quad \ldots(12)$$

$$\lambda_{B1} = \mathrm{Max}\left\{ |\alpha_1| + \frac{2\nu}{\rho\Delta\eta} - \frac{1}{2}\frac{\Delta\eta}{\Delta t}, 0.0 \right\}$$

and so on and

$$\nu = \mathrm{Max}\left\{\mu, \lambda + 2\mu, \frac{\gamma\mu}{pr}\right\}. \quad \ldots(13)$$

For regions of flow in which Δt satisfies the explicit stability criterion, all λ_{Ai} and λ_{Bi} vanish and the set of difference equation (8) just reduce to those of the explicit method.

4. Computer Program

A computer program was developed in FORTRAN for carrying out flow field calculations by the above method for transonic flow past 18% thick circular arc biconvex airfoil and was made operational on the CDC Computer of Vikram Sarabhai Space Centre, Trivandrum.

5. Initial and Boundary Conditions

The airfoil which is initially at rest is impulsively started at time zero at the desired freestream Mach number and pressure. At a sufficient distance upstream of the leading edge (in this case 8 chord lengths), the flow is assumed uniform at freestream conditions ($u = u_\infty$, $v = 0$). A similar assumption is also made in respect of the flow at the upper boundary, 6 chord lengths away. The downstream boundary is positioned far enough downstream of the trailing edge (9 chord lengths away) so that the gradients in the flow direction can be regarded as negligible $\left(\frac{\partial \phi}{\partial x} = 0\right)$. The surface of the airfoil is taken to be impermeable and no-slip boundary conditions are assumed ($u = v = 0$). This is effected by assuming a fictitious layer of mesh inside the airfoil where the values are assigned after each iteration as $u_0 = -u_1$, $v_0 = -v_1$, $p_0 = p_1$ $T_0 = T_1$ and $\rho_0 = \rho_1$. No special treatment is required for separation which will naturally develop as the marching in time progresses. The airfoil is taken to be adiabatic, and ahead and behind it, the flow is taken to be symmetric.

6. Mesh Scheme

Figure 2 desribes the mesh scheme used in the physical domain, while Figure 3 gives scheme in the computational or transformed domain.

A 40 × 30 mesh system was used in the program. Along the ξ – direction, a two-mesh system of a stretched mesh of 10 points fore and aft of the airfoil and a

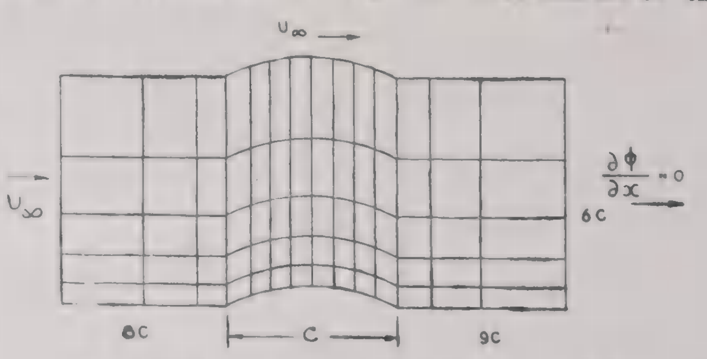

Fig. 2. Mesh about typical circular arc airfoil.

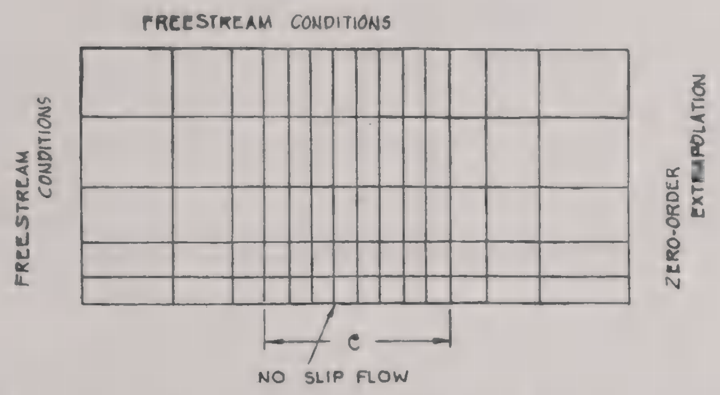

FIG. 3. Mesh system and boundary conditions in transformed computational plane.

uniform mesh of 20 points over the surface of the airfoil was applied, while in the η-direction a stretched mesh was used.

As the airfoil is thick and the flow is transonic, boundary layer separation is likely. To simulate the phenomenon reliably for turbulent flow, it is necessary to resolve the boundary layer to the sublayer scale. This sublayer scale can be taken to be nearly proportional to $1/\sqrt{Re}$ MacCarmack and Baldwin[13] so that for high Reynolds number flows of interest, the mesh resolution near the airfoil surface will be extremely fine. In the present case, we have chosen $\Delta y_{min} = \frac{1}{3} . \frac{c}{\sqrt{Re}}$, where c the chord length.

7. Turbulence Model

The well-known Cebeci-Smith's zero-equation equilibrium model of turbulence[14] was applied. Thus the eddy viscosity ϵ is evaluated using

$$\epsilon = \min (\epsilon_i, \epsilon_0) \quad \ldots(14)$$

where ϵ_i is the eddy viscosity in the inner region given by

$$\epsilon_i = 0.16y^2 \left[1 - \exp\left(- y \sqrt{\tau_w \rho} / 26\mu\right)\right] \left|\frac{\partial u}{\partial y}\right| \rho \quad \ldots(15)$$

while ϵ_0, the outer eddy viscosity, is got from

$$\epsilon_0 = 0.168\, u_{max}\, \rho \int_0^{y\delta} (1 - u/u_{max})\, dy \quad \ldots(16)$$

where

$$\tau_w = \mu_w \left(\frac{\partial u}{\partial y}\right)_{y=0} \quad \ldots(17)$$

and u_{max} is the maximum velocity in the profile extending from $y = 0$ to $y = y_\delta$.

8. Computational Details and Results

The data processing rate was about 7.6 sec per iteration, i.e. 6.3×10^{-3} sec per grid point including prediction and correction. This is, on an average, about 60% more than that for the corresponding explicit scheme. Thus when a Courant number of 5 was used, the implicit numerical scheme took about one-third of the corresponding time for the explicit method. Courant numbers upto 10 could be used without any difficulty. A maximum number of 1180 iterations were required to obtain convergence when the difference in the values of the flow parameters became less than about 0.1% (Shang and Hankey[15]) and this took about 2.5 hours on the computer. This compares very favourably with the computation time to reach steady state reported by Deiwert for similar problems on CDC 7600 computer[1]. It is found that, in the process, the soulutions have been carried out for a time corresponding to the mean flow travelling approximately 2.6 chord lengths. It is also observed that for the implicit method, the memory requirement is marginally more owing to the fact that the $\Delta\, u$'s of all elements of the state vector need to be stored. Thus, for the 40×30 mesh system used, while the explicit method requires about 3×10^4 60-bit words of core memory, an additional 6×10^3 words will have to be available for the implicit case. In other words, the increase in memory requirement for the implicit scheme is of the order of 20 per cent.

Figure 4 presents the computed pressure co-efficient variation over the surface of an 18% thick symmetric circular arc airfoil corresponding to Mach number 0.775 flow with a Reynolds number of 10^6. As can be noted, the results are in good agreement those of Deiwert[1] through an explicit integral scheme. The inviscid pressure distribution, obtained through an Euler code of the present authors[12] for the same problem, is also shown for comparison. The viscous solution lies to the left of the inviscid

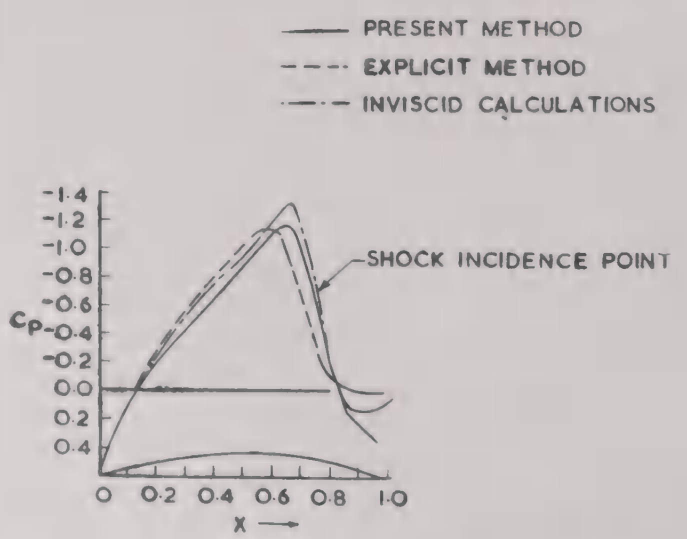

Fig. 4. Pressure coefficient over 18 percent circular arc air foil ($M_\infty = 0.775$, $R_e = 10^6$).

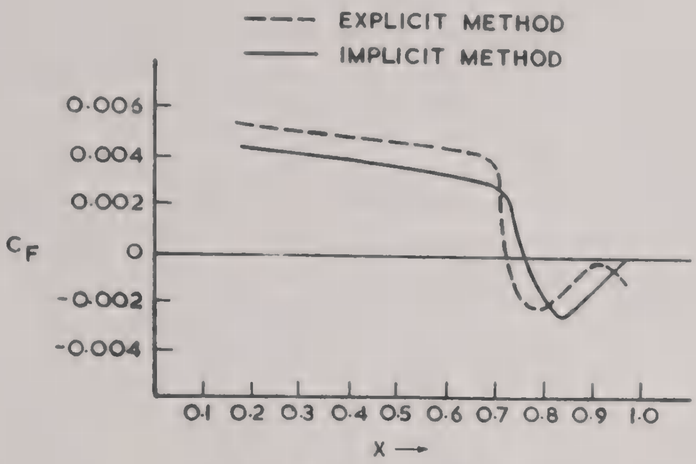

FIG. 5. Skin friction coefficient.

solution because of the boundary layer displacement effects. It is observed from the pressure distribution that the shock is centered between 73 and 78% of the chord.

Figure 5 describes the variation of the skin friction coefficient over the airfoil surface which is again in reasonable conformity with Deiwert's computed values[1], exhibited in the same figure.

9. Conclusion

An implicit numerical method, an adaptation of the well-known MacCormack scheme, for solving two-dimensional unsteady Navier-Stokes equations transformed into a rectangular computational domain is presented in this paper. The scheme is applied to simulate high Reynolds number transonic flow past 18% symmetric circular arc airfoil and a computer program developed for the purpose. The numerical results for surface pressure distribution and skin friction coefficient, obtained through the software, exhibt good agreement with the earlier computed values of Deiwert. The new method entails a substantial reduction in the computation time while retaining second-order accuracy.

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References

1. G. S. Deiwert, *AIAA Jl* 13 (1975), 1355–59.
2. G. R. Inger, *AIAA*, 82 (1982), 621–36.
3. R. W. MacCormack, *AIAA Jl.* 20 (1982), 1275–81.
4. J. S. Shang, and R. W. MacCormack, *AIAA Paper* 83-1668, 1983.
5. R. N. Gupta, P. A. Gnoffo, and R. W. MacCormack, *AIAA Jl* 23 (1985), 723–32.
6. V. Swaminathan, and N. S. Madhavan, *Proc. 3rd Int. Conf. Num. Methods for Laminar and Turbulent Flow*, Pineridge Press, Paper No. 21, 1983, 221–31.
7. N. S. Madhavan, and V. Swaminathan, *Int. Jl. Num. Methods Fluids* 6 (1986), 387–93.

8. S. L. Lawrence, J. C. Tannehill, and D. S. Chaussee, *AIAA Jl* **22** (1984), 1755–63.
9. R. W. MacCormack, *AIAA Paper* 69–354, 1984.
10. R. W. MacCormack, In : *Lecture Notes in Physics, Num. Meth Fluid Dynamics*. Springer Verlag **8** (1971), 151–63.
11. V. Swaminathan, and N. S. Madhavan, In : *Aspects of Mechanics, Lecture Notes in Applied Mathematics*, (ed. : D. K. Sinha) South Asian Pub. Co. Delhi; 1985, pp. 53–59.
12. N. S. Madhavan, and V. Swaminathan, *Indian J. Pure appl. Math.* **17** (1086), 1164–73.
13. R. W. MacCormack, and B. S. Baldwin, *AIAA Paper* 75–1, 1975.
14. T. Cebeci, A. M. O. Smith, and G. Mosinkis, *AIAA Jl* **8** (1970), 1974–82.
15. J. S. Shang, and W. L. Hankey, *AIAA Paper* 75–3, 1975.

Indian J. pure appl. Math., **19** (1) : 75–85, January 1988

FLOW IN A HELICAL PIPE

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Using an orthogonal coordinate system along a generic spatial curve, Germano[1], the problem of fully developed viscous flow in a helical pipe is studied. Assuming that the curvature κ and torsion τ are small compared to the radius a of the pipe, i. e., $\kappa a \equiv \epsilon \ll 1$, $\tau a \equiv \epsilon \lambda \ll 1$, such that $\lambda = O(1)$, the flow field is sought as a regular perturbation scheme of the main Poiseuille flow, in powers of ϵ. Analytical results obtained up to $O(\epsilon^2)$ show that the effect of torsion on helical pipe flow is a second order one, while the effect of curvature is a first order one. It is observed that the induced second order flow field due to torsion is linearly dependent on the cotangent of the helical angle of the central axis and the corresponding velocity profiles deviate symmetrically with respect to both the conjugate diameters of the circular section of the pipe. However, the flow rate remains independent of torsion upto the order considered. In the limitting case of $\lambda = 0$, the solutions of toroidal pipe flow are recovered.

1. INTRODUCTION

Curved configurations of circular tubes—such as toroidal coils, helical coils and spiral coils—are increasingly used in industrial operations involving heat exchangers, chemical reactors, rocket engines etc. Experimental investigations have shown that the flow patterns in curved tubes are significantly different over straight tubes. Dean[2] was the first person to study theoretically the flow in a curved pipe using concentric toroidal coordinate system. A secondary flow which divides itself along the diameter of the tube to constitute two sets of distinct recirculating vortices was established. Superposing the secondary flow on the motion along the channel, he showed that the resulting flow of a fluid element corresponds to a skewed helical motion. These theoretical investigations of Dean[2] were in complete agreement with dye-injection experiments of Eustice[3,4]. However, the effect of curvature was not seen in the flow rate, up to $O(\epsilon)$ considered, the parameter ϵ characterising the curvature ratio of the pipe. Subsequently Dean[5] generated higher order terms approximately, to see the said effect. Extending

Dean's work, Topakoglu[6] obtained exact solutions for the flow field up to $O(\epsilon^2)$, and significantly brought about a correction to Dean's[5] flow rate. Numerous authors later utilised Dean's coordinates to investigate the various aspects of fluid flow in toroidal pipes. But not much of a literature is available for flow in helical pipes which involve both curvature and torsion.

Wang[7] was the first person to study the effect of curvature and torsion on the flow in a helical pipe of circular cross-section. Introducing a non-orthogonal helical coordinate system and writting the continuity and Navier-Stokes equations for an incompressible viscous fluid in tensorial (Contravariant) form, Wang developed a perturbative analysis for the flow field in terms of the parameter ϵ, characterising the curvature of the central helix. His analysis is valid under the assumption that both curvature κ and torsion τ are small (i. e.)

$$\kappa a = \epsilon \ll 1, \tau a = \epsilon \lambda \ll 1$$

such that the ratio λ of torsion to curvature is of $O(1)$. The volumetric flow rate obtained by Wang shows that torsion does not affect the flow rate, to the $O(\epsilon^2)$ considered, and hence deservingly agrees with that of Topakoglu[6]. However, his secondary flow field reported upto $O(\epsilon)$ was not correct as he had failed to correlate his analysis with physical covariant description. Consequently his observation of secondary flow (for non-zero torsion) reflecting asymmetrical recirculating cells is misleading. This criticism was brought out by Germano[1]. Suitably modifying Wang's coordinate system, Germano introduced an orthogonal curvilinear coordinate system along generic spatial curve spanning the helical pipe. Deriving the governing equations of fluid flow with respect to the new frame of reference, he developed a similar analysis to that of Wang. Equations determining the first order perturbed velocity field of the main Poiseuille flow, were seen to be identical to those of Dean[2]. Hence without solving, it was possible for Germano to disprove Wang in a simple way that torsion does not cause first order effect on motion.

It is therefore reasonable to ask at what stage and how torsion can influence the flow. The analysis of Germano is extended up to $O(\epsilon^2)$ in the present paper, to provide an answer to this query.

2. Orthogonal Coordinate System and Equations

The steady, laminar flow of an incompressible viscous fluid in a helically coiled pipe, at moderately low Reynolds number is considered. The fluid transportation is caused due to pressure drop along the pipe. The central axis of the pipe, with reference to an arbitrarily chosen triad OXYZ, is described by

$$\bar{R}(s') = c \cos\left(\frac{s'}{(b^2+c^2)^{1/2}}\right)\hat{i} + c \sin\left(\frac{s'}{(b^2+c^2)^{1/2}}\right)\hat{j} + \frac{bs'}{(b^2+c^2)^{1/2}}\hat{k}. \quad \ldots(2.1)$$

s' measures the arc length along the curve, c is the radius of the circular cylinder on which the helix is coiled and $2\pi b$ is the pitch. Use of Serret-Frenet formulae gives the curvature κ and torsion τ of the central helix (2.i) as

$$\kappa = c/(b^2 + c^2), \tau = b/(b^2 + c^2). \qquad ...(2.2)$$

Let $\hat{T}$, $\hat{N}$, $\hat{B}$ denote the unit tangent, normal and binormal at a point P_1 of the central axis. The $\hat{N} - \hat{B}$ plane cuts the helical pipe in a circular section of radius a. The position vector of any point P in this section, Wang[7] is given by

$$\bar{X}(s') = \bar{R}(s') + r' \cos\theta\, \hat{N}(s') + r' \sin\theta\, \hat{B}(s') \qquad ...(2.3)$$

where r' measures the radial distance P_1P and θ the inclination of P_1P with $\hat{N}$ at P_1. Computing the metric, one can see that the coordinate system (s', r', θ) is non-orthogonal. Making use of the fact that the origion of angle θ in the plane normal to the axis is arbitrary, Germano[1] constructed an orthogonal coordinate system $(s', r', \theta + \phi(s') + \frac{1}{2}\pi)$, Fig. 1, where θ measures now the inclination of the radius

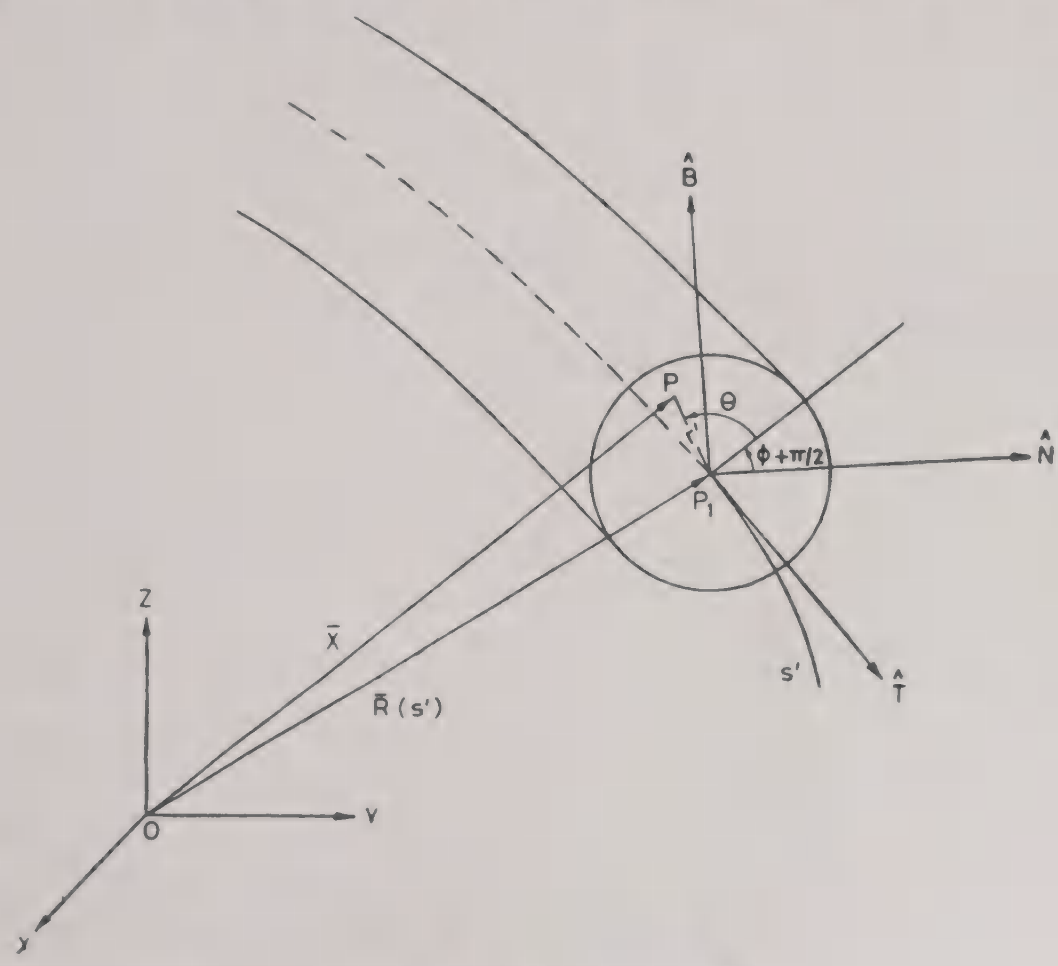

FIG. 1. Germano's coordinate system.

vector P_1P to an arbitrary direction in the normal section, the direction being characterised as inclined to the principal normal $\hat{N}$ at an angle $\frac{1}{2}\pi + \phi(s')$, with ϕ being defined by

$$\phi(s') = -\int_{s_0}^{s'} \tau(s')\, ds'. \qquad \ldots(2.4)$$

$\bar{q} = (u', v', w')$ denotes the fluid velocity at P. In terms of dimensionless variables.

$$s = s'/a,\ r = r'/a,\ u = u'/U,\ v = v'/U,\ w = w'/U,\ p = p'/\rho U^2 \qquad \ldots(2.5)$$

the equations of momentum and continuity in Germano's system are given as

$$Du + \epsilon\omega u\,[v \sin(\theta+\phi) + w \cos(\theta+\phi)] = -\omega\frac{\partial p}{\partial s}$$

$$+\frac{1}{R_e}\left[\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\left(\frac{\partial u}{\partial r}+\epsilon\omega u\sin(\theta+\phi)-\omega\frac{\partial v}{\partial s}\right)\right.$$

$$\left.+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}+\epsilon\omega u\cos(\theta+\phi)-\omega\frac{\partial w}{\partial s}\right)\right] \qquad \ldots(2.6a)$$

$$Dv - \frac{w^2}{r} - \epsilon\omega u^2\sin(\theta+\phi) = -\frac{\partial p}{\partial r}$$

$$-\frac{1}{R_e}\left[\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\epsilon\omega\cos(\theta+\phi)\right)\left(\frac{\partial w}{\partial r}+\frac{w}{r}-\frac{1}{r}\frac{\partial v}{\partial\theta}\right)\right.$$

$$\left.-\omega\frac{\partial}{\partial s}\left(\omega\frac{\partial v}{\partial s}-\frac{\partial u}{\partial r}-\epsilon\omega u\sin(\theta+\phi)\right)\right] \qquad \ldots(2.6b)$$

$$Dw + \frac{vw}{r} - \epsilon\omega u^2\cos(\theta+\phi) = \frac{-1}{r}\frac{\partial p}{\partial\theta}$$

$$+\frac{1}{R_e}\left[\left(\frac{\partial}{\partial r}+\epsilon\omega\sin(\theta+\phi)\right)\left(\frac{\partial w}{\partial r}+\frac{w}{r}-\frac{1}{r}\frac{\partial v}{\partial\theta}\right)\right.$$

$$\left.-\omega\frac{\partial}{\partial s}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}+\epsilon\omega u\cos(\theta+\phi)-\omega\frac{\partial w}{\partial s}\right)\right] \qquad \ldots(2.6c)$$

$$\omega\frac{\partial u}{\partial s}+\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial w}{\partial\theta}+\frac{v}{r}+\epsilon\omega\,[v\sin(\theta+\phi)+w\cos(\theta+\phi)] = 0 \qquad \ldots(2.6d)$$

where

$$D = \omega u\frac{\partial}{\partial s}+v\frac{\partial}{\partial r}+\frac{w}{r}\frac{\partial}{\partial\theta}$$

$$\omega = 1/[1+\epsilon r\sin(\theta+\phi)],\ R_e = Ua/\nu. \qquad \ldots(2.7)$$

The reference velocity U is the central velocity of the main Poiseuille flow.

The boundary conditions are the usual no-slip conditions on the body, viz.,

$$\bar{q} = (0,0,0) \text{ on } r = 1. \qquad \ldots(2.8)$$

3. Solutions—First Approximation

It is assumed that both curvature and torsion are small, but their ratio is appreciable i. e.

$$\kappa a = \epsilon \ll 1,\ \tau a = \epsilon\lambda \ll 1,\ \lambda = O(1). \qquad \ldots(3.1)$$

Taking the Reynolds number R_e to be of $O(1)$ and neglecting the end effects, the description of the flow through the helical pipe can be obtained as a perturbation of the main Poiseuille flow. Hence one can look for solutions describing the fully developed helical flow in the form

$$u = u_0(r) + \epsilon\, u_1(r, \alpha) + \epsilon^2 u_2(r, \alpha) + \ldots \qquad \ldots(3.2a)$$

$$v = \epsilon\, v_1(r, \alpha) + \epsilon^2 v_2(r, \alpha) + \ldots \qquad \ldots(3.2b)$$

$$w = \epsilon\, w_1(r, \alpha) + \epsilon^2 w_2(r, \alpha) + \ldots \qquad \ldots(3.2c)$$

$$p = p_0(s) + \epsilon\, p_1(r, \alpha) + \epsilon^2 p_2(r, \alpha) + \ldots \qquad \ldots(3.2d)$$

where

$$\theta + \phi = \alpha. \qquad \ldots(3.3)$$

The primary flow obtained as the zeroth order of the equations (2.6) is the well known Poiseuille flow, viz.

$$u_0(r) = 1 - r^2 \qquad \ldots(3.4a)$$

$$p_0(s) = -\frac{4}{R_e}s. \qquad \ldots(3.4b)$$

The equations determining u_1, v_1, w_1 and p_1 may be seen to be identical to those of Dean[2] and hence the corresponding solutions are

$$u_1 = U_1(r)\sin\alpha, \qquad \ldots(3.5)$$

$$v_1 = V_1(r)\sin\alpha,$$

$$w_1 = W_1(r)\cos\alpha,$$

$$p_1 = P_1(r)\sin\alpha \qquad \ldots(3.6)$$

where

$$U_1(r) = r(1 - r^2)\left[-\frac{3}{4} + \frac{R_e^2}{11520}(19 - 21r^2 + 9r^4 - r^6)\right] \qquad \ldots(3.7a)$$

$$V_1(r) = \frac{R_e}{288}(1-r^2)^2(4-r^2) \qquad \ldots(3.7b)$$

$$W_1(r) = \frac{R_e}{288}(1-r^2)(4-23r^2+7r^4) \qquad \ldots(3.7c)$$

and

$$P_1(r) = \frac{1}{12} r(9-6r^2+2r^4). \qquad \ldots(3.7d)$$

It may be noted that the solutions obtained so far are independent of λ, unlike that of Wang[7].

Hence to bring out the effect of torsion on the flow, one has to obtain higher order terms, hitherto not studied by the earlier workers.

4. Solutions—Second Approximation

The second order terms (u_2, v_2, w_2, p_2) of (3.2) are the solutions of the following set of equations:

$$v_2 \frac{du_0}{dr} - \lambda u_0 U_1 \cos\alpha + V_1 \frac{dU_1}{dr}\sin^2\alpha + \frac{U_1 W_1}{r}\cos^2\alpha$$
$$+ u_0 V_1 \sin^2\alpha + u_0 W_1 \cos^2\alpha$$
$$= \lambda P_1 \cos\alpha - r^2 \frac{dp_0}{ds}\sin^2\alpha + \frac{1}{R_e}\left[\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{\partial u_2}{\partial r}\right.$$
$$+ \frac{1}{r}\frac{\partial}{\partial\alpha}\left(\frac{1}{r}\frac{\partial u_2}{\partial\alpha}\right) + U_1\sin^2\alpha - ru_0\sin^2\alpha + \lambda V_1\cos\alpha$$
$$\left. + \frac{U_1}{r}\cos 2\alpha - u_0\cos 2\alpha - \frac{\lambda W_1}{r}\cos\alpha\right] \qquad \ldots(4.1a)$$

$$ru_0^2\sin^2\alpha - 2u_0 U_1\sin^2\alpha - \frac{W_1^2}{r}\cos^2\alpha - \lambda u_0 V_1\cos\alpha$$
$$+ V_1\frac{dV_1}{dr}\sin^2\alpha + \frac{V_1 W_1}{r}\cos^2\alpha$$
$$= -\frac{\partial p_2}{\partial r} - \frac{1}{R_e}\left[\frac{1}{r}\frac{\partial}{\partial\alpha}\left(\frac{w_2}{r} + \frac{\partial w_2}{\partial r} - \frac{1}{r}\frac{\partial v_2}{\partial\alpha}\right)\right.$$
$$\left. + \left(\frac{W_1}{r} + \frac{dW_1}{dr} - \frac{V_1}{r}\right)\cos^2\alpha - \lambda u_0\cos\alpha - \lambda\frac{dU_1}{dr}\cos\alpha\right] \ldots(4.1b)$$

$$V_1\frac{dW_1}{dr}\sin\alpha\cos\alpha + \frac{V_1 W_1}{r}\sin\alpha\cos\alpha - \frac{W_1^2}{r}\sin\alpha\cos\alpha$$

(*equation continued on p. 81*)

$$+ \lambda u_0 W_1 \sin\alpha - 2u_0 U_1 \sin\alpha\cos\alpha + r u_0^2 \sin\alpha\cos\alpha$$

$$= \frac{-1}{r}\frac{\partial p_2}{\partial \alpha} + \frac{1}{R_e}\Bigg[\frac{\partial}{\partial r}\left(\frac{w_2}{r} + \frac{\partial w_2}{\partial r} - \frac{1}{r}\frac{\partial v_2}{\partial \alpha}\right)$$

$$+ \frac{W_1}{r}\sin\alpha\cos\alpha + \frac{dW_1}{dr}\sin\alpha\cos\alpha - \frac{V_1}{r}\sin\alpha\cos\alpha$$

$$- \lambda u_0 \sin\alpha - \frac{\lambda U_1}{r}\sin\alpha\Bigg] \qquad \text{...(4.1c)}$$

$$\frac{\partial v_2}{\partial r} + \frac{v_2}{r} + \frac{1}{r}\frac{\partial w_2}{\partial \alpha} - \lambda U_1\cos\alpha + V_1\sin^2\alpha + W_1\cos^2\alpha = 0. \qquad \text{...(4.1d)}$$

These equations suggest one to look for solutions in the form

$$u_2(r,\alpha) = U_{20}(r) + U_{21}(r)\cos\alpha + U_{22}(r)\cos 2\alpha \qquad \text{...(4.2a)}$$

$$v_2(r,\alpha) = V_{20}(r) + V_{21}(r)\cos\alpha + V_{22}(r)\cos 2\alpha \qquad \text{...(4.2b)}$$

$$w_2(r,\alpha) = W_{20}(r) + W_{21}(r)\sin\alpha + W_{22}(r)\sin 2\alpha \qquad \text{...(4.2c)}$$

$$p_2(r,\alpha) = P_{20}(r) + P_{21}(r)\cos\alpha + P_{22}(r)\cos 2\alpha \qquad \text{...(4.2d)}$$

subject to the conditions that they remain finite as $r \to 0$ and vanish on $r = 1$.

The respective solutions U_{2j}, V_{2j}, W_{2j} and P_{2j} $(j = 0, 1, 2)$ are obtainable following a straight forward sequential procedure, though involving voluminous algebra.

The consolidated results for the velocity and pressure fields up to $O(\epsilon^2)$ are given below.

$$u = (1 - r^2)\Bigg[1 + \epsilon r\left\{-\frac{3}{4} + \frac{R_e^2}{11520}(19 - 21r^2 + 9r^4 - r^6)\right\}\sin\alpha$$

$$+ \epsilon^2\Bigg\{-\frac{1}{32}(3 - 11r^2) - \frac{R_e^2}{230400}(148 + 43r^2 - 132r^4$$

$$+ 68r^6 - 7r^8) - \frac{R_e^4}{3715891200}(4119 - 17161r^2 + 29179r^4$$

$$- 26261r^6 + 13569r^8 - 4015r^{10} + 605r^{12} - 35r^{14}) + \lambda r\Bigg(\frac{R_e}{576}(29$$

$$+ 5r^2 - 3r^4) + \frac{R_e^3}{29030400}(2969 - 4381r^2 + 3249r^{14}$$

$$- 1301r^6 + 274r^8 - 20r^{10})\Bigg)\cos\alpha + r^2\left(-\frac{5}{16} + \frac{R_e^2}{276480}\right.$$

(equation continued on p. 82)

$$(463 - 613r^2 + 296r^4 - 40r^6) - \frac{R_e^4}{117050572800}(145690$$

$$- 240206r^2 + 174649r^4 - 70547r^6 + 19123r^8 - 2801r^{10}$$

$$\left. \left. + 160r^{12}) \Big) \cos 2\alpha \right\} \right] + O(\epsilon^3) \qquad \ldots(4.5a)$$

$$v = (1 - r^2)^2 \left[\epsilon \frac{R_e}{288} (4 - r^2) \sin \alpha + \epsilon^2 \left\{ - \frac{R_e}{576} r (4 - r^2) \right. \right.$$

$$+ \lambda \left(\frac{1}{6} + \frac{R_e^2}{69120} (13 - 15r^2 + 7r^4 - r^6) \right) \cos \alpha$$

$$+ r \left(\frac{R_e}{240} (3 - r^2) - \frac{R_e^3}{232243200} (4979 - 2792r^2 + 777r^4 \right.$$

$$\left. \left. \left. - 134r^6 + 5r^8) \right) \cos 2\alpha \right\} \right] + O(\epsilon^3) \qquad \ldots(4.5b)$$

$$w = (1 - r^2) \left[\epsilon \frac{R_e}{288} (4 - 23r^2 + 7r^4) \cos \alpha + \epsilon^2 \left\{ \lambda \left(- \frac{1}{12} \right. \right. \right.$$

$$\left. (2 - r^2) - \frac{R_e^2}{69120} (13 - 224r^2 + 266r^4 - 124r^6 + 17r^8) \right) \sin \alpha$$

$$+ r \left(\frac{-R_e}{960} (12 - 59r^2 + 21r^4) + \frac{R_e^3}{232243200} (4979 - 20521r^2 \right.$$

$$\left. \left. \left. + 13499r^4 - 4421r^6 + 829r^8 - 35r^{10}) \right) \sin 2\alpha \right\} \right] + O(\epsilon^3)$$

$$\ldots(4.5c)$$

$$p = - \frac{4}{R_e} s + \epsilon \frac{r}{12} (9 - 6r^2 + 2r^4) \sin \alpha + \epsilon^2 \left[- \frac{r^2}{144} (81 \right.$$

$$- 81r^2 + 28r^4) + \frac{R_e^2}{1382400} r^2 (1140 - 1095r^2 + 200r^4$$

$$+ 225r^6 - 108r^8 + 15r^{10});$$

$$+ \lambda r \left(- \frac{1}{6R_e} (1 - 3r^2) + \frac{R_e}{17280} (101 - 120r^2 + 90r^4 \right.$$

$$\left. - 30r^6 + 3r^8) \right) \cos \alpha + r^2 \left(\frac{1}{120} (54 - 55r^2 + 20r^4) \right.$$

$$- \frac{R_e^2}{3870720} (3597 - 8344r^2 + 9240r^4 - 5040r^6 + 1288r^8$$

$$\left. \left. - 120r^{10}) \right) \cos 2\alpha \right] + O(\epsilon^3). \qquad \ldots(4.5d)$$

5. Flow Rate

The volume rate of discharge of the fluid through the circular cross section of the pipe is given in terms of dimensionless variables as

$$\frac{\tilde{q}}{Ua^2} = \int_0^{2\pi}\int_0^1 ur \, d\alpha \, dr \qquad \ldots(5.1)$$

Computing the integral (5.1), one obtains the flow rate $Q = \tilde{q}/2\pi\, Ua^2$ as

$$\frac{Q}{Q_s} = 1 - \frac{\epsilon^2}{48}\left[\frac{1541}{67200}\left(\frac{R_e}{6}\right)^4 + \frac{11}{10}\left(\frac{R_e}{6}\right)^2 - 1\right] + O(\epsilon^3) \qquad \ldots(5.2)$$

where Q_s is the repective flow rate in a straight tube.

6. Discussion

The flow rate (5.2) obtained comparatively with lesser effort using Germanos'[1] Orthogonal Coordinate system, agrees identically with Wang's[7] result. The terms involving λ in the expression (4.5a) for u are periodic in α, hence the reason for the flow rate (5.1) to be independent of torsion. Naturally the expression (5.2) agrees with the result of Topakoglu[6] for a toroidal pipe. Owing to enhanced mixing and momentum transfer due to secondary flows, it is common belief that the total frictional loss of energy near the wall increases and consequently the fluid experiences more resistance in passing through the circular/helical pipe. Looking at this angle, in the limit of small ϵ, it can be seen from the analysis that when $R_e > 5.67$, the flux in the helical pipe is less than that of a straight pipe. However for the flow to remain laminar, a rough estimation yields an upper bound for R_e as $R_e \approx 40/\sqrt{\epsilon}$. The corresponding Dean number $\left(2\epsilon R_e^2\right)$ is approximately 3200.

The computed flow fleld (4.5), enables one to verify analytically that the effect of torsion on a helical pipe flow is a second order one, while the effect of curvature is a first order one. It can be observed that the induced second order flow is linearly dependent on the cotangent of the helical angle of the central generic spatial curve. As the terms involving λ are all periodic in α, the presence of torsion causes a symmetric deviation of the velocity profiles, not only with respect to the diameter of full circle circle symmetry (a diameter of the circular section containing the principal normal) but also with respect to the conjugate diameter. However the full profiles of the velocities remain symmetric only with respect to the diameter of the full circle symmetry. It can be seen that the flow field (4.5) reported in this paper is finer and exact upto the order considered compared to the approximated results of Murata *et al.*[8]. By putting $\lambda = 0$ one can recover the flow field of Topakoglu[6], for toroidal pipe flow.

If terms of $O(\epsilon^2)$ are taken into consideration, in view of non-zero torsion it is not possible to express the secondary velocities in terms of a stream function and con-

sequently project the streamline flow pattern. However, confining to $O(\epsilon)$, one can express the secondary velocities in the (r, θ) plane in terms of stream function ψ as

$$\frac{\psi}{R_e} = -\frac{1}{288}(4r - 9r^3 + 6r^5 - r^7)\cos\alpha \qquad \ldots(6.1)$$

where

$$v_1 = \frac{1}{r}\frac{\partial\psi}{\partial\alpha}, \; w_1 = -\frac{\partial\psi}{\partial r}. \qquad \ldots(6.2)$$

The expression (6.1) is independent of λ, thereby showing (as observed earlier) that torsion has no first order effect on the flow. This disproves Wang[7], who erroneously calculated the secondary flows and predicted asymmetric recirculating cells which tend to coalesce with decreasing R_e, for non-zero torsion. This was due to the fact that Wang[7] failed to correlate the contravariant components of the velocity vector to a physical covariant description, as remarked by Germano[1]. Fig. 2 projects the streamlines

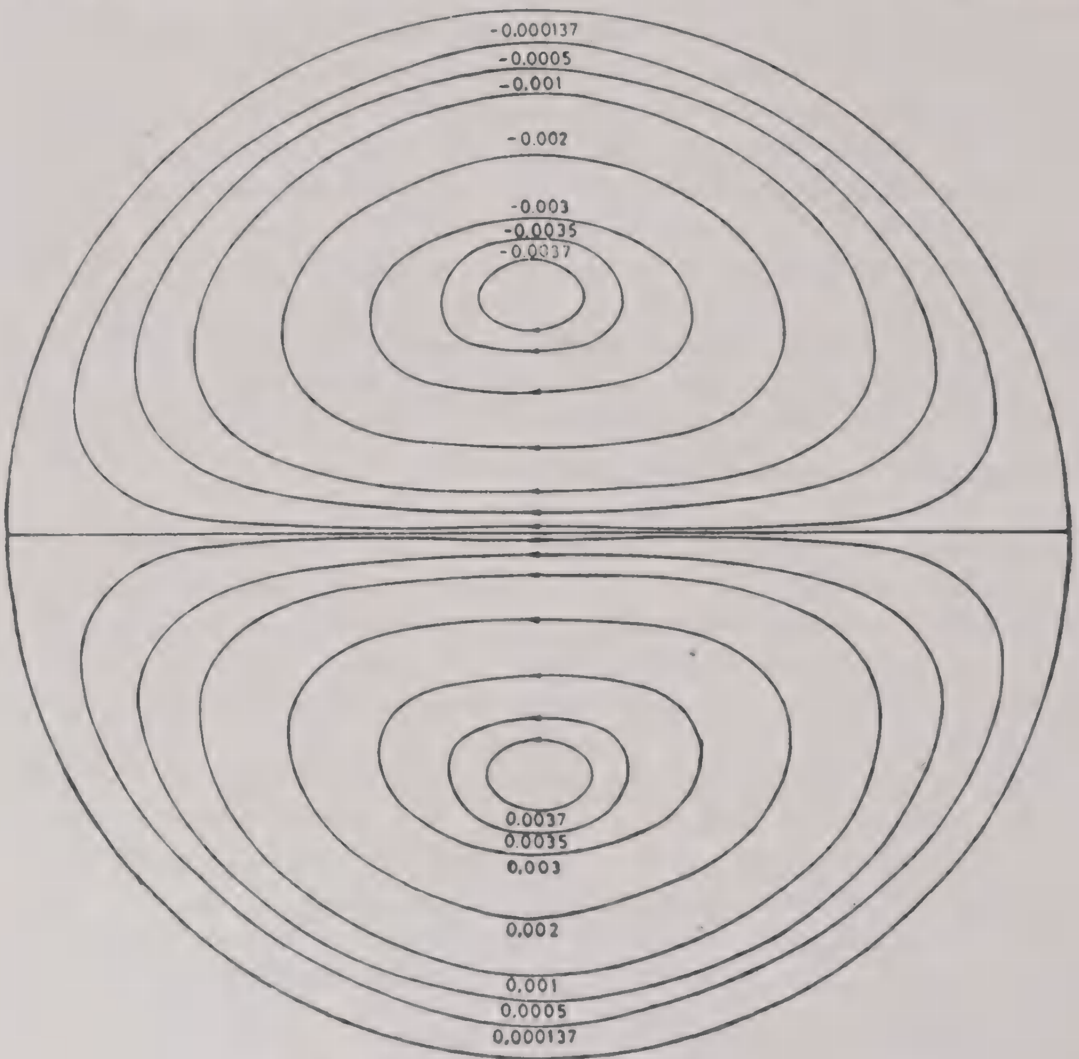

FIG. 2. Secondary streamlines. ψ/R_e=constant.

$\psi \pm 0.0037, \pm 0.0035, \pm 0.0030, \pm 0.0020, \pm 0.0010, \pm 0.0005, \pm 0.000137$ in the (r, θ) plane. It can be seen easily from the diagram that the secondary flow field divides itself along the diametral axis of the section of the pipe into two mirror images of recirculating cells. Obviously these streamlines are the same as that of Wang[7], for the case $\lambda = 0$. The vortex centre (point of no secondary flow) observed at $r = 0.43$, $\alpha = \pi/2$, though well away from the origin, is still closer to the diameter of the full circle symmetry than the upper wall. As expected, this observation is in agreement with Dean[2] and Austin and Seader[9].

REFEYENCES

1. M. Germano, *J. Fluid Mech.* **1** (1982), 125.
2. W. R. Dean, *Phil. Mag.* 4 (7) (1927), 208.
3. J. Eustice, *Proc. R. Soc.* A84 (1910), 107.
4. J. Eustice, *Proc. R. Soc.* A85 (1911), 119.
5. W. R. Dean, *Phil. Mag.* 5 (7) (1928), 673.
6. H. C. Topakoglu, *J. Math. Mech.* **16** (1967), 1321.
7. C. Y. Wang, *J. Fluid Mech.* **108** (1981), 185.
8. S. Murata, Y. Miyake, and T. Inaba, *Bull. JSME* **24** (1981), 355.
9. L. R. Austin, and J. D. Seader, *A. I. Ch. E. J.* **19** (1973), 85.

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AN ALGEBRAICALLY SPECIAL BIANCHI TYPE VI_h COSMOLOGICAL MODEL IN GENERAL RELATIVITY

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In this paper a Bianchi type VI_h space-time filled with a viscous fluid is considered which is of Petrov type II. Various physical properties have also been discussed.

1. Introduction

Bianchi type VI_h universes representing perfect fluid distribution have been studied by Collins[2] and Wainwright and Anderson[5]. It is well known that such universes do not go over to FRW universes asymptotically but they can be made as close to such models as one wants them to be in a finite interval of time, the latter corresponding either to the very early stage of the universe or a late or suitably intermediate stage in the evolution of the actual universe. In the early stage of the universe matter behaved like a viscous fluid during the period of neutrino-decoupling. It is also expected that during the big bang explosion copious amount of gravitational radiation was produced[1].

In this paper, we have derived a Bianchi type VI_h universe which is filled with a viscous fluid and which is of Petrov type II, the latter condition ensuring that the universe contains gravitational radiation. The model so constructed gives rise to two different cases, in each of which the expanding and the contracting phases of the universe are described separately by two different line-elements joining smoothly at the point of no expansion. In each case the universe expands from an initial singularity till the expansion stops and thereafter contracts and collapses into a final singularity. The model in some special situations can be regarded as close to FRW universe during the early stages of its evolution. The relative behaviour of electric and magnetic parts of the free gravitational field near both the singularities have been studied and it is found that in one case the gravitational field is of type N near the final singularity.

2. Derivation of the Line-element

We have taken the line-element which describes Bianchi type VI_h space-times in the form

$$ds^2 = -dt^2 + A^2(t)\,dx^2 + B^2(t)\,e^{2x}\,dy^2 + C^2(t)\,e^{2hx}\,dz^2 \qquad ...(2.1)$$

where $h \neq \pm 1$.

We assume that the universe is filled with a viscous fluid, the energy-momentum tensor T_{ij} of which is given by

$$T_{ij} = (\epsilon + \bar{p})\, v_i\, v_j + \bar{p} g_{ij} - 2\eta\sigma_{ij}$$

with

$$\bar{p} = p - \zeta\theta;$$

p, ϵ, η, ζ, θ, σ_{ij} and v_i being respectively the pressure, density, coefficients of shear and bulk viscosities, expansion scalar, shear tensor and the unit flow vector of the fluid assumed to be orthogonal to the hyersurfaces of homogeneity.

The field equations

$$R_i^j - \tfrac{1}{2} R g_i^j + \Lambda g_i^j = -8\pi T_i^j$$

lead to

$$\left.\begin{aligned} -8\pi\bar{p} - \Lambda &= \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} - \tfrac{2}{3}\eta\left(2\frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) - \frac{h}{A^2} \\ &= \frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{C}\dot{A}}{CA} - \tfrac{2}{3}\eta\left(-\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) - \frac{h^2}{A^2} \\ &= \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \tfrac{2}{3}\eta\left(-\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + 2\frac{\dot{C}}{C}\right) - \frac{1}{A^2} \end{aligned}\right\} \quad \ldots(2.2)$$

$$8\pi\epsilon - \Lambda = \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} - \frac{h^2 + h + 1}{A^2} \quad \ldots(2.3)$$

$$0 = -(1 + h)\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + h\frac{\dot{C}}{C} \quad \ldots(2.4)$$

an overhead dot standing for ordinary differentiation with respect to t.

From (2.4), we get

$$A^2 = K^2 \mu\, \nu^{1-h/1+h} \quad \ldots(2.5)$$

where $\mu = BC$, $\nu = B/C$ and K is a constant, Equations (2.2) and (2.5) lead to the single equation

$$\frac{d}{dt}\left(\frac{\dot{\nu}}{\nu}\right) + \frac{\dot{\nu}}{2\nu}\left(3\frac{\dot{\mu}}{\mu} + \frac{1-h}{1+h}\cdot\frac{\dot{\nu}}{\nu}\right) + \frac{h^2 - 1}{A^2} = -16\pi\eta\frac{\dot{\nu}}{\nu} \quad \ldots(2.6)$$

in three unknowns μ, ν and η. In order to get a determinate solution we assume the following :

(1) The metric (2.1) is of Petrov type II,

(2) η is proportional to $| \theta |$.

Assumption (1) leads to

$$\frac{d}{dt}\left(\frac{\dot{\nu}}{\nu}\right)+\frac{\dot{\nu}}{2\nu}\left(\frac{\dot{\mu}}{\mu}-\frac{1-h}{1+h}\cdot\frac{\dot{\nu}}{\nu}\right)-\frac{h^2-1}{A^2}=\frac{4h}{(1+h)}\cdot\frac{1}{A}\cdot\frac{\dot{\nu}}{\nu} \quad \text{...(2.7)}$$

while (2) gives

$$\eta = M\theta \text{ for } \theta > 0 \quad \text{...(2.8a)}$$

and

$$\eta = -M\theta \text{ for } \theta < 0. \quad \text{...(2.8b)}$$

M being a positive dimensionless constant. Equations (2.6) and (2.7) under the condition (2.8a) give the following solution :

$$A = L_1 \cdot \frac{\{(1+h)^2+(1-h)^2 e^{qT}\}^{\alpha/2} \cdot e^{1/2\beta_1 qT}}{\{f(1-e^{qT})\}^{1/2(\alpha+\beta_1+\gamma_1)}} \quad \text{...(2.9a)}$$

$$B = L_2 \cdot \frac{(1+h)^2+(1-h)^2 e^{qT}\}^{\alpha/2} \cdot e^{1/2\beta_2 qT}}{\{f(1-e^{qT})\}^{1/2(\alpha+\beta_2+\gamma_2)}} \quad \text{...(2.9b)}$$

$$C = L_3 \cdot \frac{\{(1+h)^2+(1-h)^2 e^{qT}\}^{\alpha/2} \cdot e^{1/2\beta_3 qT}}{\{f(1-e^{qT})\}^{1/2(\alpha+\beta_3+\gamma_3)}} \quad \text{...(2.9c)}$$

in which $f = +1$ or -1 according as $h > -1$ or $h < -1$, L_i are arbitrary constants and the constants q, α, β_i and γ_i are given by

$$q = -\frac{2(1+h^2)}{(1+h)}, \quad \alpha = -\frac{2}{(1+24\pi M)}$$

$$\beta_1 = \frac{(1+8\pi M)(1+h)^2-2h}{(1+h^2)(1+24\pi M)}, \quad \beta_2 = \frac{8\pi M(1+2h)(1+h)^2+(1+3h^2)}{(1-h)(1+h^2)(1+24\pi M)}$$

$$\beta_3 = \frac{-8\pi M(h+2)(1+h)^2-h(3+h^2)}{(1-h)(1+h^2)(1+24\pi M)}, \quad \gamma_1 = \frac{(1+8\pi M)(1-h)^2+2h}{(1+h^2)(1+24\pi M)}$$

$$\gamma_2 = \frac{8\pi M(1+2h)(1-h)+(1+h)}{(1+h^2)(1+24\pi M)}, \quad \gamma_3 = \frac{-8\pi M(h+2)(1-h)+h(1+h)}{(1+h^2)(1+24\pi M)}.$$

The same set of equations under the condition (2.8b) gives

$$A = \bar{L}_1 \cdot \frac{\{(1+h)^2+(1-h)^2 e^{qT}\}^{\bar{\alpha}/2} \cdot e^{1/2\bar{\beta}_1 qT}}{\{f(1-e^{qT})\}^{1/2(\bar{\alpha}+\bar{\beta}_1+\bar{\gamma}_1)}}. \quad \text{...(2.10a)}$$

$$B = \bar{L}_2 \,.\, \frac{\{(1+h)^2 + (1-h)^2\, e^{qT}\}^{\bar{\alpha}/2} \,.\, e^{1/2\bar{\beta}_i qT}}{\{f(1-e^{qT})\}^{1/2(\bar{\alpha}+\bar{\beta}_2+\bar{\gamma}_2)}} \qquad \ldots(2.10b)$$

$$C = \bar{L}_3 \,.\, \frac{\{(1+h)^2 + (1-h)^2\, e^{qT}\}^{\bar{\alpha}/2} \,.\, e^{1/2\bar{\beta}_3 qT}}{\{f(1-e^{qT})\}^{1/2(\bar{\alpha}+\bar{\beta}_3+\bar{\gamma}_3)}} \qquad \ldots(2.10c)$$

in which $\bar{L}_i$ are arbitrary constants and the constants $\bar{\alpha}$, $\bar{\beta}_i$ and $\bar{\gamma}_i$ are given by the corresponding unbarred contstants of the solution (2.9) when M is replaced by $-M$. T in both the solutions (2.9) and (2.10), is related to the cosmic time t by $t = \int A dT$.

The line-element (2.1) corresponding to both the solutions (2.9) and (2.10) has singularities at $T = 0$ and $T = \pm\infty$. The expression for scalar of expansion θ corresponding to solution (2.9) is given by

$$\theta = \frac{f\, e^{-1/2\beta_1 qT}}{L_1(1+h)(1+24\pi M)} \cdot \frac{\{f(1-e^{qT})\}^{\left(\frac{\alpha+\beta_1+\gamma_1}{2}-1\right)}}{\{(1+h)^2 + (1-h)^2\, e^{qT}\}^{(\alpha/2+1)}}$$
$$\times\ [(1+h^2-h-H) + (1+h^2+h+H)\, e^{qT}]$$
$$\times\ [(1+h^2-h+H) + (1+h^2+h-H)\, e^{qT}]$$

where $H = \sqrt{3(1+h^2+h^4)}$. Similar expression can be written for the expansion scalar $\bar{\theta}$ corresponding to the solution (2.10). We assume that $(1-24\pi M) > 0$. From the expressions for θ and $\bar{\theta}$ we find that they vanish at $T=T'$ and $T=T''$, where

$$T' = \frac{1}{q}\log\left(\frac{H-h^2+h-1}{H+h^2+h+1}\right) \text{ and } T'' = \frac{1}{q}\log\left(\frac{H+h^2-h+1}{H-h^2-h-1}\right).$$

When $h > -1$, the model of the universe is described by the line-element (2.1) corresponding to solutions (2.9) and (2.10) in the time-ranges $0 < T < T'$ and $T > T'$ respectively. Similarly, for $h < -1$, it is described by the solutions (2.9) and (2.10) respectively in the time-spans $0<T<T''$ and $T>T''$. The line-elements corresponding to both the solutions (2.9) and (2.10) join smoothly at the common boundary $T=T'$ or T'' in the two cases mentioned above. This requires[4] that across the surface of discontinuity $x^4 \equiv T = T'$ or T'', the following are continuous : A, B, C, dA/dT, dB/dT, dC/dT and T_1^4. These conditions are found to be satisfied provided

$$\bar{L}_i = L_i \,.\, \frac{\{(1+h)^2 + (1-h)^2\, e^{qT}\}^{\frac{\alpha-\bar{\alpha}}{2}} \,.\, e^{1/2q(\beta_i-\bar{\beta}_i)\tau}}{\{f(1-e^{q\tau}\}^{1/2(\alpha+\beta_i+\gamma_i-\bar{\alpha}-\bar{\beta}_i-\bar{\gamma}_i)}}$$

where τ stands for T' or T'' and $i = 1, 2, 3$. For $h > -1$, the model evolves with a big bang at $T = 0$ and expands till $T = T'$. Thereafter it contracts till it collapses at $T = +\infty$. Similarly for $h < -1$, the model evolves with a big bang at $T = 0$,

expands till $T = T''$ and after that contracts to collapse at $T = +\infty$. In terms of the cosmic time t, the initial singularity occurs at $t = 0$ and the collapse occurs after a finite lapse of time in each case considered above. The types of the initial singularity are restricted to those of cigar and infinite pan cake of first and second kind[3], while those of the final singularity are restricted to point, barrel and cigar depending on the values of h and M.

The anisotropy σ/θ is finite at the start. It increases for $h \geqslant 0$ and $h < -1$, but for $-1 < h < 0$, it decreases and attains its minimum at $T = \bar{T}$, where

$$\bar{T} = \frac{1}{q} \log \left[\frac{(1+h)^2 (h^2 - h + 1)}{(1-h)^2 (h^2 + h + 1)}\right].$$

By a suitable choice of h the anisotropy can be made to be small in the interval $(\bar{T} - \delta, \bar{T} + \delta)$ for a given $\delta > 0$ so that the model is approximately FRW in this interval. We also find that the electric part of the Weyl tensor is dominant over its magnetic part except at the collapse when $h > -1$. In the latter case the model is asymptotically Petrov type N. The condition $\epsilon > 0$ requires necessarily that $h < 0$ and

$$M < \left[\frac{1}{24\pi} \cdot \left\{\frac{(1-h)}{(1+h+h^2)^{1/2}} - 1\right\}\right] \text{ when } \Lambda = 0.$$

All the above discussions hold good for the solution corresponding to perfect fluid when $M = 0$. In this case the reality condition $-\epsilon < p \leqslant \epsilon$ is satisfied for $-1 < h < 0$ together with a suiable choice of Λ.

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References

1. B. J. Carr, *Astr. Ap.* 89 (1980), 6.
2. C. B. Collins, *Commun. Math. Phys.* 23 (1971), 137.
3. S. R. Roy, S. Narain, and J. P. Singh, *Aust. J. Phys.* 38 (1985), 239.
4. J. L. Synge, *Relativity, the General Theory*, North Holland Publishing Co. Amsterdam, 1960, p. 39.
5. J. Wainwright, and P. J. Anderson, *Gen. Rel. Grav.* 16 (1984), 609.

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ON THE ORBITS IN THE LIE ALGEBRAS OF SOME (PSEUDO) ORTHOGONAL GROUPS

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A complete classification of the orbits in the Lie algebras of all the real orthogonal and pseudo-orthogonal groups of total dimension not exceeding five is presented. The classification is carried out using elementary geometrical methods, exhibiting in a clear way the relevance of the results for a lower dimensional group in obtaining the results for a higher dimensional one. For each orbit the values of the algebraic invariants are calculated, a representative element is displayed, and the geometric nature of the latter is described by listing a complete set of independent vectors invariant under it. While the orbit structure for the orthogonal groups turns out to be relatively simple, that for the Lorentz type and the de Sitter type pseudo-orthogonal groups become progressively complex. Particular care has been taken, in view of the intricacy of many of the results, to develop a suggestive and systematic notation. The orbits are classified and tabulated in a form that makes it particularly easy to apply them in practical physical problems. Examples of such problems are pointed out.

1. INTRODUCTION

The real orthogonal and pseudo-orthogonal groups of low dimensions play an important role in a variety of problems of physical interest. Thus, for instance, problems possesing spherial spmmetry in three dimensions involve the group SO (3) in their analysis[1,2]. Physical systems subject to the requirements of special relativity similarly involve the (homogeneous orthochronous) Lorentz group SO (3, 1), and in suitable kinematical situations also the important subgroups SO (3) and SO (2.1)[1,2]. The latter subgroup, SO (2,1), is closely related to the two-dimensional real unimodular group SL (2,R) which is the same as the real symplectic two-dimensional group Sp (2, R) relevant in Hamiltonian mechanics[1,2]. Thus SL (2, R) is the group of linear canonical transformations on one canonical pair of Hamiltonian variables; and as is well-known, there is a two-to-one homomorphism from SL (2, R) to SO (2,1). Similarly, when one considers problems involving two canonical pairs of variables on a four dimensional phase space, the group of linear canonical transformations is the symplectic group Sp (4, R), which

is a double covering of the real pseudo-orthogonal de Sitter group SO (3, 2) in five dimensions. Physical problems in which SO (2, 1) and SO(3, 2) play a significant role on account of their relation to canonical transformations are many, among which we mention here the following as examples : Fourier optics in the paraxial limit and the related study of ideal optical systems[3-8]; description and propagation of optical Gaussian Schell-model beams[9,10]; squeezed coherent states[11-15] and two-photon coherent states[16-18] in quantum optics; the representation theory of para-Bose operator algebras[19-28] and studies of particles with internal structure[29] based on the new Dirac equation[30,31]. Of course the relevance of the de Sitter groups SO (3,2) and SO (4, 1) in the context of certain linear relativistic quantum mechanical wave equations has been long appreciated; thus the tensor and vector matrices associated with the original Dirac equation[32] generate a de Sitter algebra, and a corresponding statement is true in the case of the infinite component Majorana equations [33-35] as well as with the well-known Bhabha equations[36].

For most practical physical applications, it is adequate and convenient to work initially with the elements of the Lie algebras of these groups, and later by a process of integration or exponentiation to arrive at finite group elements. This is particularly true in dealing with the linear (unitary or nonunitary) representations of these groups. In the case of the simplest group SO (3), it is a well-known and geometrically evident fact that all (infinitesimal) generators are basically alike, differing from one another only in orientation and overall magnitude. This is the essential content of Euler's theorem[37] which states that every rotation in three dimensions leaves one direction invariant, and so is a rotation through some angle about that direction as axis. However, when one goes to higher dimensions, or alters the signature of the metric, or both, the elements of the Lie algebra separate into many essentially distinct types, with quite different geometrical properties. This situation can be expressed in the following way: The Lie algebra **G** of any one of the groups G under consideration, viewed as a linear vector space, carries a particular representation of G, namely the adjoint representation. Two vectors in the Lie algebra, i. e. two infinitesimal generators, which are connected by some transformation in the adjoint representation may be regarded as being essentially equivalent and not differing from one another in any intrinsic manner. Starting with any element in the Lie algebra and subjecting it to all the transformations of the adjoint representation, one builds up the orbit on which the starting element lies. The entire Lie algebra **G** thus splits into distinct and mutually disjoint orbits under the adjoint action. (Of course all the preceding statements are valid for any Lie group, not just the ones we are concerned with here).. While for the group SO(3) all orbits in its Lie algebra **SO (3)** (except the trivial one) are basically similar in structure, this is not so in the other cases, and one does find significantly different kinds of elements and therefore of orbits in the Lie algebra. Examples of this situation are of course familiar in the context of special relativity, where generators of spatial rotations and of pure Lorentz transformations are the opposite ends of a spectrum of possibilities.

It is the purpose of this paper to provide a complete classification of the orbits in the Lie algebras of all the real orthogonal and pseudo-orthogonal groups of total dimension not exceeding five. Our aim is to make use of elementary geometrical methods in obiaining this classification, and also to exhibit in the clearest possible way the relevance of the the results for a lower dimensional groups in obtaining the results for a higher dimensional one. We endeavour to derive and present our results in a manner that makes it particularly easy to apply them in practical physical problems. We shall be concerned with the orthogonal groups SO (n) for $n = 3, 4, 5$; the Lorentz pseudo-orthogonal groups SO $(n, 1)$ for $n = 2, 3, 4$; and the "de Sitter" type pseudo-othogonal groups SO $(n,2)$ for $n = 2, 3$. Thus there are eight groups included in our study, divided in the above manner into three distinct sets. In some cases, such as SO(3) and possibly also SO (2, 1), the classification of and the geometrical nature of the elements on each orbit is well known. Nevertheless, for the sake of completeness and the setting up of uniform notations, we shall include all cases in the analysis, the familiar ones being dealt with only briefly.

The material of this paper is organised as follows. In section 2 the three groups SO (n) are taken up, in the sequence $n = 3, 4,5$. For the treatment of **SO (4)**, the decompositions SO (4) $\simeq$ SO (3) $\otimes$ SO (3) (locally) and **SO (4) = SO (3) $\oplus$ SO (3)** are exploited. Section 3 treats the Lorentz type grous SO$(n, 1)$ for $n = 2,3$ and 4. In the last of these, namely in classifying the orbits in **SO (4,1)**, it is necessary in one case to deal with an E (3) subgroup of SO (4, 1), and its Lie algebra. Section 4 classifies orbits in the two "de Sitter" type algebras **SO** (2,2), **SO** (3,2). For the former, the decomposition **SO (2.2) = SO (2,1) $\otimes$ SO (2,1)** in exploited. Since the number of different types is quite large in these two cases, the results are presented in two separate tables (corresponding to ranks 2 and 4 respectively) in each case. The **SO(3,2)** analysis involves, in a particular situation, use of an **E (2,1)** subalgebra. The paper concludes in section 5 with some general comments.

As mentioned earlier, in order to make the results more transparent and useful and to clarify the relationships between the structures for different groups, we will express the orbit classifications for the various groups in a mutually compatible manner. This means that the notation, in particular the choices of indices labelling components of vectors, tensors, .. and their ranges, must be chosen judiciously. We now explain the choices which we shall adhere to throughout the paper. For the two groups SO(3), SO (2,1) operating on three-dimensional spaces, we use lower case Latin letters a, b, c, ... as indices for components of vectors, tensors, etc. For the three groups SO(4), SO (3.1), SO (2.2) operating on four-dimensional spaces the lower case Greek letters λ, μ, ν, ... will be used. For the three groups SO (5), SO (4,1), (SO (3,2) acting on five-dimensional spaces, the capitan Latin letters A, B, C, ... will be used. Turning to the ranges of indices for the orthogonal groups SO(n) the dimensions will be numbered 1, 2, ... 3. Thus for SO (3) the indices a, b, ... run over 1, 2, 3; for SO (4) the indices λ, μ, ... go from 1 to 4; and for SO (5), A, B, ... run over 1, 2, ..., 5. For these three

groups, the metric tensor is just the kronecker symbol δ_{ab}, $\delta_{\mu\nu}$ or δ_{AB}. For the Lorentz type groups SO $(n, 1)$ the dimensions will be numbered 0, 1, 2, 3, 4. The metric tensor $g_{..}$ will be diagonal and "space like": $g_{00} = -1$, $g_{11} = \ldots g_{44} = 1$. In the discussion of SO (2,1), we will let $a, b, ..$ run over 0, 1, 2; for SO (3,1) we shall have $\mu, \nu, \ldots$ going over 0, 1, 2, 3; and for SO (4, 1), $A, B, \ldots$ will span the full range 0, 1, .., 4. For the "de Sitter" type groups SO $(n, 2)$ we number the dimensions as 0, 1, 2, 3, 5 omitting the numeral 4. This is in fact the convention often used in physical problems where SO (3,2) in relevant. The metric tensor will again be diagonal and "space like": $g_{00} = g_{55} = -1$, $g_{11} = g_{22} = g_{33} = 1$. (There is a minor mismatch here in that the dimension 5 carries a positive signature in the context of SO (5) but a negative signature in the case of SO $(n, 2)$; however this will not cause any serious problem). For SO (2,2) we let $\mu, \nu, \ldots$ go over 0, 1, 2, 5; and for SO (3,2) we have $A, B, \ldots$ going over the full range 0, 1, 2, 3, 5. With these conventions, the appearance af indices $a, b, \ldots$ will immediately signify that we are dealing with "three dimensional quantities"; whether the relevant group is SO (3) or SO (2, 1) will be clear from the context. Similarly the appearance of indices $\mu, \nu, \ldots$ will signify that "four-dimensional objects" are involved, and so on. The generic symbol $J_{..}$ will be a basis element for any one of the Lie algebras : thus J_{ab} for SO (3) and SO (2,1); $J_{\mu\nu}$ for SO (4), SO (3,1) and SO (2,2); and J_{AB} for SO (5), SO (4,1) and SO (3,2). In all cases we have antisymmetry in the subscripts. The components of a general element in the Lie algebra will be $\xi^{..}$, with antisymmetry in the superscripts. Thus the Lie algebra element will be

$$J(\xi) = \tfrac{1}{2}\, \xi^{ab} J_{ab} \text{ or } \tfrac{1}{2}\, \xi^{\mu\nu} J_{\mu\nu} \text{ or } \tfrac{1}{2}\xi^{AB} J_{AB} \qquad \ldots(1.1)$$

with all indices being lowered in the case of SO (n). The quadratic invariant will uniformly be denoted by $\mathscr{C}_1(\xi)$

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\, \xi^{ab} \xi_{ab} \text{ or } \tfrac{1}{2}\, \xi^{\mu\nu} \xi_{\mu\nu} \text{ or } \tfrac{1}{2}\xi^{AB} \xi_{AB}. \qquad \ldots(1.2)$$

For the cases of groups in four or five dimensions, there is a second algebraic invariant $\mathscr{C}_2(\xi)$ which will be defined at the appropriate places. It is easily constructed once one realises that, in all cases, the adjoint representation transforms $\xi^{..}$ as a second rank antisymmetric tensor under the appropriate orthogonal or pseudo-orthogonal rotation group. Finally the symbols e_a e_μ, e_A will be used for a basic set of mutually orthogonal unit vectors in three, four or five dimensions respectively; while the letters t, l and s (relevant only for the SO $(n, 1)$ and SO $(n, 2)$ analyses) will stand for general "timelike" "lightlike" and "spacelike" vectors.

2. Orbits in the Lie Algebras SO (n) $n, = 3, 4, 5$

The generators J_{ab} of SO (3) obey the familiar Lie bracket relations

$$[J_{ab}, J_{cd}] = \delta_{ac} J_{bd} - \delta_{bc} J_{ad} + \delta_{ad} J_{cb} - \delta_{cd} J_{ca}, \; a, b, \ldots = 1,2,3. \qquad \ldots(2.1)$$

The Lie relations for SO (4) and SO (5) (indeed, for any SO (n)) are similar, with a, b, c, d replaced by μ, ν, ρ, σ or A, B, C, D and the ranges of the indices suitably extended:

they need not be written down explicitly. We now briefly review the orbit structure in **SO** (3), then take up the cases of **SO (4)** and **SO (5)**.

SO (3) A general element in the Lie algebra is as in eqn. (1.1)

$$J(\xi) = \tfrac{1}{2}\, \xi_{ab}\, J_{ab}. \qquad \ldots(2.2)$$

With the use of the three-index antisymmetric symbol, both J_{ab} and ξ_{ab} can be replaced by single index "vector" quantities

$$J_a = \tfrac{1}{2}\, \epsilon_{abc}\, J_{bc},\ \xi_a = \tfrac{1}{2}\, \epsilon_{abc}\, \xi_{bc};$$

$$J_{ab} = \epsilon_{abc}\, J_c,\ \xi_{ab} = \epsilon_{abc}\, \xi_c;$$

$$J(\xi) = \xi_a\, J_a. \qquad \ldots(2.3)$$

We treat ξ_a as the components of a three-vector ξ. In terms of J_a, eqn. (2.1) takes the familar from

$$[J_a, J_b] = \epsilon_{abc}\, J_c. \qquad \ldots(2.4)$$

The quadratic invariant $\mathscr{C}_1(\xi)$, the only one in the case of **SO** (3), is the squared length of ξ:

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\, \xi_{ab}\, \xi_{ab} = \xi_a\, \xi_a = |\, \xi\, |^2 \qquad \ldots(2.5)$$

If $\delta\theta$ is a small parameter, the effect on a general vector z_a of an infinitesimal transformation generated by $J(\xi)$ is

$$\delta\, z_a = \delta\theta\, \xi_{ab}\, z_b = -\,\delta\theta\, (\xi_\wedge\, \mathbf{z})a. \qquad \ldots(2.6)$$

Thus ξ itself is invariant under the rotations generated by $J(\xi)$. This can be understood as a matrix property which we later generalise to higher dimensional groups. The three-dimensional real antisymmetric matrix (ξ_{ab}) is necessarily of rank 2, since we exclude $\xi = 0$; it therefore has exactly one null eigenvector, namely ξ itself, which is therefore invariant under the rotations generated by $J(\xi)$[38].

The adjoint action of **SO** (3) on ξ, as is well known, amounts to subjecting ξ to a three-dimensional rotation. It is conveniently represented via the spin 1/2 representation of **SO (3)**, which also leads to the defining representation of SU (2). In it, the generators J_a are the Pauli matrices :

$$J_a \to \frac{-i}{2}\, \sigma_a, \quad a = 1,2,3. \qquad \ldots(2.7)$$

$J(\xi)$ is a general traceless antihermitian 2×2 matrix :

$$J(\xi) = \frac{-i}{2}\, \xi.\, \sigma. \qquad \ldots(2.8)$$

For any $U \in$ SU (2), the adjoint action changes ξ to ξ' in this way :

$$U\, J(\xi)\, U^{-1} = J(\xi'),$$

$$\xi'_a = R_{ab}(U)\, \xi_b. \qquad \ldots(2.9)$$

Here R(U) $\in$ SO(3) is the image of U $\in$ SU (2) under the homomorphism SU (2) $\rightarrow$ SU (3). Thus the orbit of ξ consists of all ξ' with the same (squared) length as ξ. We can therefore label obits in **SO(3)** with a positive nonzero parameter u: the orbit $\vartheta_3(u)$ consists of all $J(\xi)$ for which

$$\mathscr{C}_1(\xi) = |\xi|^2 = u^2. \qquad \text{...(2.10)}$$

This is a sphere S^2 in the 3-dimsnsional ξ space. A convenient representative element on $\vartheta_3(u)$ is the positive multiple uJ_{12} of J_{12}. This element can be characterized geometrically by the statement that under the rotatious generated by it, the single vector e_3 is invariant. This reflects the fact that the rank of the matrix (ξ) is constant over an orbit, and so is 2 at the representative point uJ_{12}. All these properties for **SO** (3) can be summarised in a table which sets the pattern for presentation of results in other cases:

SO(3) Orbit structure :

Rank (ξ)	Orbit	Parameter range	Invariant $\mathscr{C}_1(\xi)$	Representative Point	Invariant vector
2	$\vartheta_3(u)$	$u > 0$	u^2	uJ_{12}	e_3

SO (4)

With the index conventions explained in the Introduction, a general element of **SO** (4) is written as

$$J(\xi) = \tfrac{1}{2}\,\xi_{\mu\nu}\,J_{\mu\nu} \qquad \text{...(2.11)}$$

the subscripts taking the values 1,..., 4. It is convenient to define three-component objects and quantities in the following manner :

$$J_a = \tfrac{1}{2}\,\epsilon_{abc}\,J_{bc}, \quad K_a = J_{4a};$$
$$\xi_a = \tfrac{1}{2}\,\epsilon_{abc}\,\xi_{bc}, \quad \eta_a = \xi_{4a}. \qquad \text{...(2.12)}$$

(Of course, the latin subcripts here go over 1, 2, 3). Then the basic Lie relations of **SO(4)** are :

$$[J_a, J_b] = [K_a, K_b] = \epsilon_{abc}\,J_c,$$
$$[J_a, K_b] = \epsilon_{abc}\,K_c. \qquad \text{...(2.13)}$$

If now we define the combinations

$$M_a = \tfrac{1}{2}(J_a + K_a)$$
$$N_a = \tfrac{1}{2}(J_a - K_a) \qquad \text{...(2.14)}$$

the familiar **SO** (3) $\oplus$ **SO** (3) structure of **SO** (4) emerges :

$$[M_a, M_b] = \epsilon_{abc}\,M_c,$$
$$[N_a, N_b] = \epsilon_{abc}\,N_c,$$
$$[M_a, N_b] = 0. \qquad \text{...(2.15)}$$

Therefore the results of the orbit classificatian for **SO (3)** can be used to tackle the **SO (4)** problem.

The general element $J(\xi) \in$ **SO** (4) can be written in the forms

$$J(\xi) = \xi.\,\mathbf{J} + \eta.\,\mathbf{K}$$

$$= \alpha.\,\mathbf{M} + \beta.\,\mathbf{N},$$

$$\alpha_a = \xi_a + \eta_a, \quad \beta_a = \xi_a - \eta_a. \qquad \text{...(2.16)}$$

The invariant $\mathscr{C}_1(\xi)$ has the value

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\,\xi_{\mu\nu}\,\xi_{\mu\nu} = |\,\xi\,|^2 + |\,\eta\,|^2 = \tfrac{1}{2}(\,|\,\alpha\,|^2 + |\,\beta\,|^2). \qquad \text{...(2.17)}$$

Since SO (4) is a group of rank two (as is SO (5)), there is a second algebraic invariant which can be obtained with the use of the four index antisymmetric symbol :

$$\mathscr{C}_2(\xi) = -\tfrac{1}{8}\,\epsilon_{\mu\nu\rho\sigma}\,\xi_{\mu\nu}\,\xi_{\rho\sigma}$$

$$= \xi.\,\eta = \tfrac{1}{4}(\,|\,\alpha\,|^2 - |\,\beta\,|^2). \qquad \text{...(2.18)}$$

This invariant is related to the rank of the 4×4 antisymmetric matrix $(\xi_{\mu\nu})$: in fact one finds

$$\Delta(\xi) = \det(\xi_{\mu\nu}) = (\xi.\,\eta)^2$$

$$= (\mathscr{C}_2(\xi))^2. \qquad \text{...(2.19)}$$

Now the rank of $(\xi_{\mu\nu})$ is either 2 or 4,since we exclude $\xi_{\mu\nu} = 0$. Therefore the vanishing of $\mathscr{C}_2(\xi)$ corresponds to rank $(\xi_{\mu\nu}) = 2$, and a nonvanishing $\mathscr{C}_2(\xi)$ implies rank $(\xi_{\mu\nu})=4$. The rank in turn determines the number of independent vectors in four-space invariant under the infinitesimal rotations generated by $J(\xi)$. On a general four-vector z_μ, this rotation acts as

$$\delta z_\mu = \delta\theta\,\xi_{\mu\nu}\,z_\nu. \qquad \text{...(2.20)}$$

We see: if $\mathscr{C}_2 = 0$, there are two independent vectors which are both invariant under the rotations (2.20), and without loss of generality they may be assumed to be orthonormal ; if $\mathscr{C}_2 \neq 0$, there are no such vectors.

On account of the local SO (3) $\otimes$ SO (3) structure of SO (4), the effect of the adjoint action is to subject the two three-component quantities α_a, β_a to independent SO (3) rotations. With this remark and the results of the **SO (3)** analysis, we can immediately classify the **SO (4)** orbits. Given an element $J(\xi) \in$ **SO** (4), there is a unique element on its orbit having the form $|\,\alpha\,|\,M_3 + |\,\beta\,|\,N_3$. Here $|\,\alpha\,|$ and $|\,\beta\,|$ cannot both vanish. $J(\xi)$, and with it all the elements on its orbit, can be characterised as being of rank 2 if $|\,\alpha\,| = |\,\beta\,|$; otherwise they are all of rank 4. Rewritten in terms of the original $J_{\mu\nu}$, the above mentioned representative element is[39].

$$|\,\alpha\,|\,M_3 + |\,\beta\,|\,N_3 = u\,J_{12} + u'\,J_{43},$$

$$u = \tfrac{1}{2}(\,|\alpha| + |\beta|\,) > 0,$$

$$u' = \tfrac{1}{2}(\,|\alpha| - |\beta|\,),\ u \geqslant |u'|\,. \qquad \text{...(2.21)}$$

Now, the rank 2 case corresponds to vanishing u'. For such orbits we see that $u > 0$ is a single labelling parameter, and the choice of representative element is such that e_3 and e_4 are both invariant under the rotations generated by it. On the other hand, the rank 4 case corresponds to $u' \neq 0$, and the rotations generated by $uJ_{12} + u'J_{43}$ definitely alter every non-zero four-vector. These results can be summarised as follows :

SO (4) Orbit structure :

Rank (ξ)	Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Invariant $\mathscr{C}_2(\xi)$	Representative Point	Invariant vectors
2	$\vartheta_4(u)$	$u > 0$	u^2	0	uJ_{12}	e_3, e_4
4	$\vartheta_4(u, u')$	$u \geqslant |u'| > 0$	$u^2 + u'^2$	uu'	$uJ_{12}+u'J_{43}$	—

The following remarks can now be made concerning these results. The problem of classifying the rank 2 orbits of **SO (4)** reduces easily to a problem at the level of the **SO** (3) which generates the canonical SO (3) subgroup in SO (4) acting on the dimensions 1, 2 and 3. This is because on any such orbit one can always find representative elements for which one of the invariant vectors is e_4. On restricting oneself to this part of the orbit, the further classification depends only on the already available **SO** (3) results. On the other hand, the rank 4 orbits in **SO (4)** are quite new in the sense that they cannot be reduced to a problem within the **SO (3)** algebra corresponding to the canonical SO(3) subgroup of SO (4); they may be thought of as characteristic of SO (4), notwithstanding the fact that the local SO (3) $\otimes$ SO (3) structure of SO (4) simplified matters. The table of results for **SO (4)** also shows that the values of the algebraic invariants $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ together determine the orbit to which $J(\xi)$ belongs. (Note that they are restricted by $\mathscr{C}_1 \geqslant 2\,|\,\mathscr{C}_2\,|$). In the rank 2 case, when $\mathscr{C}_2 = 0$, $\mathscr{C}_1$ determines u and the statement follows. In the rank 4 case, $\mathscr{C}_1$ and $\mathscr{C}_2$ do determine u and u' individually because of thc restriction $u \geqslant |\,u'\,|$, and the statement again follows.

SO(5)

A general element of **SO** (5) is

$$J(\xi) = \tfrac{1}{2}\,\xi_{AB}\,J_{AB}, \qquad \text{...(2.22)}$$

with the indices going over 1, 2, ..., 5. The infinitesimal SO (5) rotations produced by this generator alter a general vector z_A in the standard manner:

$$\delta z_A = \delta\theta\,\xi_{AB}\,z_B. \qquad \text{...(2.23)}$$

The first algebraic invariant is

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\,\xi_{AB}\,\xi_{AB}. \qquad \text{...(2.24)}$$

The rank of the matrix (ξ_{AB}) is either 2 or 4; correspondingly the number of independent null vectors of this matrix, equivalently vectors invariant under (2.23), is either 3 or 1. Thus for any $J(\xi)$ there is at least one invariant vector. In constructing the second algebraic invariant $\mathscr{C}_2(\xi)$ we are led to a possible invariant vector defined in terms of ξ_{AB} itself. Since with SO (5) we have a five-index antisymmetric symbol, we define the five-vector

$$\zeta_A = \tfrac{1}{8}\,\epsilon_{ABCDE}\,\xi_{BC}\,\xi_{DE}. \qquad \ldots(2.25)$$

It is an easily checked fact that

$$\xi_{AB}\,\zeta_B = 0. \qquad \ldots(2.26)$$

Therefore, whenever ζ_A does not vanish identically, it provides us with one vector invariant under (2.23). The second algebraic invariant is the squared length of ζ :

$$\mathscr{C}_2(\xi) = \zeta_A\,\zeta_A. \qquad \ldots(2.27)$$

In the case when rank $(\xi_{AB}) = 2$, let us denote a preliminary choice of three independent null vectors of (ξ_{AB}) by $e^{(m)}_A$ $m = 1, 2, 3$, and set up the matrix of inner products

$$(M_{mm'}) = (e^{(m)}_A\;\; e^{(m')}_A) = (e^{(m)}.\, e^{(m')}). \qquad \ldots(2.28)$$

This three-dimensional real symmetric matrix is positive definite because we are dealing with SO (5) rather than SO (4, 1) or SO (3,2). Now by an SO (3) transformation acting on the indices m, m', and therefore amounting to a different choice of the $e^{(m)}$, we can diagonalize M, when its nonzero entries become all strictly positive. By a further renormalization of its eigenvetors, it can be seen that M becomes the unit matrix. This argument shows that without loss of generality the three invariant vectors under the transformation (2.23) can be chosen to be othonormal[40].

With the help of the above result, the analysis of rank 2 orbits in SO (5) becomes quite easy. Let some $J(\xi) \in$ SO(5) of rank 2 be given. By means of suitable SO (5) transformations we can pass to those elements on the orbit of $J(\xi)$ for which the three invariant vectors are e_3, e_4 and e_5. Such elements must be of the form uJ_{12}, $u \neq 0$, where $|u|$ is fixed by the value of $\mathscr{C}_1(\xi)$. This is similar to the SO (3) situation. Since the elements uJ_{12} and $-uJ_{12}$ can be connected to one another even within SO (3), it follows that we can restrict u to be strictly positive in choosing uJ_{12} as an orbit representative for rank 2 orbits of SO(5). On the other hand, no further reduction in distinct orbit representatives is possible even with the greater freedom of transformation available with SO(5) as compared with SO (3); i. e. as one can quite easily convince oneself, two elements uJ_{12} and $u'J_{12}$ with $u, u' > 0$, $u \neq u'$, cannot be connected to one another by any SO (5) transformation. At the representative point uJ_{12}, the only nonzero component of ξ_{AB} is $\xi_{12} = u$; so $\zeta_A = 0$ identically at this point and consequently also at every point

on every rank 2 orbit. This result may in a sense have been anticipated : if ζ_A were not identically zero, it "would not know which of the three independent invariant vectors" it should be. As with SO (3) and SO (4) the present rank 2 orbits will be denoted as $\vartheta_5(u)$.

The classification of rank 4 orbits is slightly more intricate. Let an element $J(\xi) \in$ SO (5) of rank 4 be given. The matrix (ξ_{AB}) has just one nonzero null vector. Using suitable SO (5) transformations we can pass to elements on the orbit of $J(\xi)$ for which the invariant vector is e_5. Such elements therefore belong to **SO(4)**, the Lie algebra of the canonical SO (4) subgroup of SO (5) acting on the dimensions 1, 2, 3, 4; and moreover they are of rank 4 within **SO (4)**, meaning that there is nonontrivial combination e_1, e_2, e_3, e_4 annihilated by any of them. Now from the **SO (4)** results we know that with the help of transformations within SO (4) we can find among the above elements on the orbit of $J(\xi)$ some of the form $uJ_{12} + u'J_{43}$ with $u \geqslant |u'| > 0$. However, since an SO (5) orbit could be larger than an SO (4) orbit, two distinct elements of the above form which cannot be connected within SO (4) may possibly be connected by some SO (5) transformation. If this happens, the concerned SO (5) transformation must take e_5 into $-e_5$. The point being made is that while in the first instance, by arranging the invariant vector to be e_5, the problem is brought down to the level of rank 4 **SO (4)** orbits, in thereafter using the **SO (4)** results one must take account of the fact that SO (5) is larger than SO(4). In this way one sees that, say by a rotation of amount π in the 4-5 plane, the elements $uJ_{12} + u J_{43}$ and $uJ_{12} - u'J_{43}$ are on the same **SO** (5) orbit. Thus unlike the **SO** (4) case, we may here restrict the parameters by $u \geqslant u' > 0$; the corresponding orbit in **SO (5)**, denoted $\vartheta_5(u, u')$, is of rank 4 with unique representative element $uJ_{12} + u'J_{43}$. At this point on the orbit, only ξ_{12} and ξ_{43} out of ξ_{AB} are non-zero; correspondingly, $\zeta_5 = -uu'$ is non zero, while the other components $\zeta_\mu = 0$. This is consistent with our arranging the choice of orbit representative so that the invariant vector is e_5. The general conclusion to be drawn is that at every point $J(\xi)$ on a rank 4 orbit, the five-vector ζ_A does not vanish, and is the single vector annihilated by $J(\xi)$. The final results for **SO (5)** are thus as follows :

SO (5) Orbit structure :

Rank (ξ)	Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Invariant $\mathscr{C}_2(\xi)$	Representative Point	Invariant vectors
2	$\vartheta_5(u)$	$u > 0$	u^2	0	uJ_{12}	e_3, e_4, e_5
4	$\vartheta_5(u,u')$	$u \geqslant u' > 0$	$u^2 + u'^2$	$u^2 u'^2$	$uJ_{12} + u'J_{43}$	e_5

Several points are worth noting. All results for **SO (5)** are obtainable on making use of the previously obtained results for **SO (3)** and **SO (4)**, provided one pays attention to the extra freedom of transformation available within SO (5) as compared to SO (4).

As with SO (4), the vanishing here of the algebraic invariant $\mathscr{C}_2(\xi)$ unambiguously signifies that the rank is 2. The restriction on $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ in the **SO (5)** case are:

$$\mathscr{C}_1 > 0,\ \mathscr{C}_2 \geqslant 0,\ \mathscr{C}_1 \geqslant 2\sqrt{\mathscr{C}_2}. \qquad \ldots(2.29)$$

We see that $\mathscr{C}_1(\xi)$ determines u in the rank 2 case, while $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ determine u, u' unambiguously in the rank 4 case. Thus the values of the algebriac invariants fix the orbit to which $J(\xi)$ belongs. The manner in which the **SO** (4) results helped simplify the **SO** (5) problem sets the pattern for similar simplifications in the **SO** (n, 1) and **SO** (n, 2) analyses.

3. Orbits in the Lie Algebras SO (n, 1), $n = 2, 3, 4$

A basis for **SO (2,1)** is given by three elements $J_{ab} = -J_{ba}$, $a, b = 0, 1, 2$, obeying the bracket relations

$$[J_{ab}, J_{cd}] = g_{ac}\, J_{bd} - g_{bc}\, J_{ad} + g_{ad}\, J_{cb} - g_{bd}\, J_{ca}. \qquad \ldots(3.1)$$

The diagonal metric is $g_{00} = -1$, $g_{11} = g_{22} = 1$. For SO (3, 1) and SO (4,1) we replace a, b, c, d by μ, ν, ρ, σ and A, B, C, D respectively, and extend the metric tensor with $g_{33} = g_{44} = 1$. For the pseudo-orthogonal groups indices must be raised and lowered using the appropriate metric tensor, and the antisymmetric symbol is defined by

$$\epsilon_{012} = \epsilon_{0123} = \epsilon_{01234} = 1. \qquad \ldots(3.2)$$

SO (2,1)

A general element of **SO (2,1)** is

$$J(\xi) = \tfrac{1}{2}\,\xi^{ab}\, J_{ab}. \qquad (3.3)$$

Since the dimension of the space is three, as with **SO** (3) we can use ϵ_{abc} to replace J_{ab} and ξ_{ab} by single index objects:

$$J_a = \tfrac{1}{2}\,\epsilon_{abc}\, J^{bc}, \quad \xi_a = -\tfrac{1}{2}\,\epsilon_{abc}\,\xi^{cb}\ ;$$
$$J_{ab} = -\,\epsilon_{abc}\, J^c, \quad \xi_{ab} = \epsilon_{abc}\,\xi^c. \qquad \ldots(3.4)$$

The relative signs are adjusted so that $J(\xi)$ has a neat form :

$$J(\xi) = \xi^a J_a. \qquad \ldots(3.5)$$

In terms of J_a, the bracket relations (3.1) are

$$[J_a, J_b] = \epsilon_{abc}\, J_c. \qquad \ldots(3.6)$$

Under the adjoint action by SO (2,1), when ξ^{ab} transforms as a second rank antisymmetric tensor, ξ^a transforms as a three-vector because ϵ_{abc} is an invariant tensor. The quadratic invariant $\mathscr{C}_1(\xi)$, the only algebraic invariant in the **SO(2,1)** case, is

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\,\xi^{ab}\,\xi_{ab} = -\,\xi^a\,\xi_a. \qquad \ldots(3.7)$$

The rank of the matrix (ξ_{ab}) must be 2, which means that under infinitesimal Lorentz transformations

$$\delta z^a = \delta\theta \xi^a_b z^b \qquad \ldots(3.8)$$

there is just one invariant vector. This is ξ^a itself since

$$\xi^a_b \ \xi^b = 0. \qquad \ldots(3.9)$$

As long as $J(\xi)$ is non-zero, so is ξ^a; and $J(\xi)$ generates Lorentz transformations "about ξ^a as axis". The adjoint action can be explicitly realised via the 2×2 matrix representation of SL (2, R), as in the SO (3) case. In this representation we have, for instance.

$$J_0 \to \frac{i}{2}\sigma_2,\ J_1 \to \tfrac{1}{2}\sigma_1,\ J_2 \to \tfrac{1}{2}\sigma_3 \qquad \ldots(3.10)$$

so $J(\xi)$ is a general real traceless 2×2 matrix :

$$J(\xi) = \tfrac{1}{2}(\xi^0 . i\sigma_2 + \xi^1 \sigma_1 + \xi^2 \sigma_3). \qquad \ldots(3.11)$$

Then for $S \in SL(2, R)$ we have

$$SJ(\xi) S^{-1} = J(\xi')$$

$$\xi'^a = \Lambda^a{}_b(S)\, \xi^b, \qquad \ldots(3.12)$$

where $\Lambda(S) \in SO(2, 1)$ is the image of $S \in SL(2, R)$ under the homomorphism $SL(2, R) \to SO(2,1)$.

We see from this discussion that, since there are three qualitatively different kinds of vectors in a 2+1 space, there is a similar number of qualitatively different orbit types. If $J(\xi)$ with a timelike ξ^a is given, the orbit of $J(\xi)$ consists of all $J(\xi')$ with ξ'^a having the same Lorentz square as ξ^a, and ξ'^0 the same sign as ξ^0. Similar statements can be made for the cases when ξ^a is lightlike (positive or negative). When ξ^a is spacelike only the same Lorentz square is required. Thus each nontrivial orbit can be distinguished by a symbol t, l or s in these three cases. Further, in the t case, the representative point on the orbit can be chosen as $\xi'^a = (u, 0, 0)$, $u \neq 0$; in the l case we can arrange $\xi'^a = \epsilon(1, 0, 1)$, $\epsilon = \pm 1$; and in the s case, $\xi'^a = (0, 0, v)$, $v > 0$. At these representative points the invariant vectors are respectively e_0, $e_0 + e_2$ and e_2.

A table presenting all the distinct orbits in SO (2,1) can be drawn up based on these results :

SO (2.1) orbit structure ;

Rank (ξ)	Orbit	Parameter range	Invariant $\mathscr{C}_1(\xi)$	Representative Point	Invariant vector
2	$\vartheta_{2,1}(t; u)$	$u \neq 0$	u^2	uJ_{12}	e_0
2	$\vartheta_{2,1}(l; \epsilon)$	$\epsilon = \pm 1$	0	$\epsilon(J_{12} + J_{10})$	$e_0 + e_2$
2	$\vartheta_2(s; v)$	$v > 0$	$-v^2$	vJ_{10}	e_2

The "arguments" of $\vartheta_{2,1}$ (...) in the various cases are self-explanatory. In comparison with the **SO (3)** table, there is an expected increase in complexity. In particular we see that only when $\mathscr{C}_1(\xi)$ is negative does it uniquely determine the orbit to which $J(\xi)$ belongs. In case $\mathscr{C}_1(\xi) > 0$, this algebraic invariant cannot determine the sign of u or ϵ, as the case may be.

SO (3,1)

In handling this case we can follow the SO (4) pattern to the extent possible and deal with three-dimensional quantities by breaking up $\xi_{\mu\nu}$ and $J_{\mu\nu}$; we can also exploit the simplicity of the defining two-dimensional spinor representation of SL (2,C), as was done with SO (3) and SO (2,1). In the sequel both approaches will be used and related to each other. Concerning the index conventions, since in this Section dealing with the SO $(n, 1)$ groups the Latin indices a, b, ...run over 0, 1, 2, we shall use indices j, k, ... to go over the range 1, 2, 3 covering the "space" dimensions

In terms of $\xi_{\mu\nu}$ and $J_{\mu\nu}$ we define

$$J_j = \tfrac{1}{2}\,\epsilon_{jkl}\,J_{kl},\ K_j = J_{0j};$$
$$\xi_j = \tfrac{1}{2}\,\epsilon_{jkl}\,\xi_{kl},\ \eta_j = \xi_{j0}. \qquad ..(3.13)$$

Then the general element $J(\xi) \in$ **SO (3,1)** and the Lie brackets are

$$J(\xi) = \tfrac{1}{2}\,\xi^{\mu\nu}\,J_{\mu\nu} = \boldsymbol{\xi}.\mathbf{J} + \boldsymbol{\eta}.\mathbf{K};$$
$$[J_j, J_k] = -[K_j, K_k] = \epsilon_{jkl}\,J_l,$$
$$[J_j, K_k] = \epsilon_{jkl}\,K_l. \qquad ...(3.14)$$

The two algebraic invariants are

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\,\xi^{\mu\nu}\,\xi_{\mu\nu} = |\,\boldsymbol{\xi}\,|^2 - |\,\boldsymbol{\eta}\,|^2,$$
$$\mathscr{C}_2(\xi) = \tfrac{1}{8}\,\epsilon_{\mu\nu\rho\sigma}\,\xi^{\mu\nu}\,\xi^{\rho\sigma} = \boldsymbol{\xi}.\,\boldsymbol{\eta}. \qquad ...(3.15)$$

The infinitesimal Lorentz transformation generated by $J(\xi)$:

$$\delta z^\mu = \delta\theta.\xi^\mu{}_\nu.z^\nu \qquad ...(3.16)$$

is characterised by the 4×4 antisymmetric matrix $(\xi_{\mu\nu})$ whose rank is either 2 or 4. The rank can be related to $\mathscr{C}_2(\xi)$ since

$$\Delta(\xi) = \det(\xi_{\mu\nu}) = (\boldsymbol{\xi}.\boldsymbol{\eta})^2 = (\mathscr{C}_2(\xi))^2. \qquad ...(3.17)$$

Thus vanishing $\mathscr{C}_2(\xi)$ means rank $(\xi_{\mu\nu}) = 2$, and there are then two independent vectors invariant under (3.16): nonvanishing $\mathscr{C}_2(\xi)$ means rank $(\xi_{\mu\nu}) = 4$, hence no invariant vectors.

At this point we introduce the two-dimensional spinor representation of $J_{\mu\nu}$:

$$J_j = \frac{-i}{2}\,\sigma_j,\ K_j = -\frac{1}{2}\,\sigma_j;$$ (*equation continued on p.* 104)

$$J(\xi) = -\frac{i}{2}(\xi - i\,\eta).\sigma. \qquad \ldots(3.18)$$

So $J(\xi)$ is a general complex traceless 2×2 matrix. Adjoint action by SL (2,C), which is the same as by SO (3, 1), amounts to subjecting $J(\xi)$ to a similarity transformation: for any $S \in$ SL (2, C),

$$S(\xi - i\eta).\sigma\, S^{-1} = (\xi' - i\eta').\sigma \qquad \ldots(3.19)$$

In searching for the "most natural" form into which $J(\xi)$ can be put via the adjoint representation, we can therefore use the results of the theory of the Jordan canonical form of a matrix. Remembering the tracelessness property, this allows for two possibilities: (i) $J(\xi)$ can be diagonalised, with nonzero equal and opposite generally complex eigenvalues; (ii) $J(\xi)$ cannot be diagonalised, but can be put into the upper triangular Jordan from $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. These two distinct possibilites can be partially correlated with the classification based on the rank of $(\xi_{\mu\nu})$, since in the spinor representation

$$\det J(\xi) = -\frac{1}{4}\det(\xi - i\eta).\,\sigma = \frac{i}{2}\mathscr{C}_2(\xi) - \frac{1}{4}\mathscr{C}_1(\xi). \qquad \ldots(3.20)$$

Therefore possibility (i) with diagonalisable $J(\xi)$ must correspond to at least one of $\mathscr{C}_1$ and $\mathscr{C}_2$ being nonzero; while possibility (ii) implies $\mathscr{C}_1 = \mathscr{C}_2 = 0$. Only in the latter case can the definite statement be made that rank $(\xi_{\mu\nu}) = 2$;in the former case, we can have both values 2 and 4 for the rank, according as $\mathscr{C}_2 = 0$ or $\neq 0$.

With this preparation, we can proceed to analyse and classify first the rank 2 orbits in **SO (3,1)**. Let $J(\xi)$ be given with $\mathscr{C}_2(\xi) = 0$. By adapting the argument given in Section 2 in connection with SO (5), we can assume without loss of generality that the two independent null-vectors of $(\xi_{\mu\nu})$ are mutually orthogonal, and each of them is normalised to $\pm$ 1 if it is not a light-like vector. In a $1 + 3$ space with signature $-+++$ there are three distinct possibilities for this pair of vectors : is *ts*, *ls* or *ss*. Throughout an orbit one and the same possibility is realised. There is therefore at least one spacelike vector invariant under $J(\xi)$. Using suitable SO (3,1) transformations, we can pass to those elements on the orbit of $J(\xi)$, each of which leaves e_3 invariant. Such elements therefore are linear combinations of J_{ab}, $a, b = 0, 1, 2$, and to further analyse them the **SO (2,1)** results can be used. These now tell us that one of the following three mutually exclusive possibilities must occur : (i) there are elements uJ_{12}, $u \neq 0$, on the orbit of $J(\xi)$, for which e_0, e_3 are the two invariant vectors, realising the configuration *ts*; (ii) there are elements $\epsilon\,(J_{12} + J_{10})$, $\epsilon = \pm 1$, on the orbit of $J(\xi)$, for which $e_0 + e_2$, e_3 are two invariant vectors, corresponding to the configuration *ls*; (iii) there is an element vJ_{10}, $v > 0$, on the orbit of $J(\xi)$, which leaves e_2, e_3 invariant and realises the configuration *ss*. Obviously in case (i) the value of $\mathscr{C}_1(\xi)$ fixes $|u|$, and we can use the extra freedom available in SO (3,1) as compared to SO (2, 1) to achieve $u > 0$; in case (ii) both $\mathscr{C}_1$ and $\mathscr{C}_2$ vanish, and in case $\epsilon = -1$ it can be converted to $+1$ by a rotation of amount π in the $1 - 3$ plane; in case (iii) $\mathscr{C}_1(\xi)$ determines $v > 0$ unam-

biguously. Of these three cases, we can recognise that case (ii) is precisely that possibility encountered among the Jordan canonical forms when $J(\xi)$ could not be diagonalised. Since in cases (i) and (iii) $\mathcal{C}_1(\xi)$ is nonzero, these are included among those Jordan canonical forms wherein $J(\xi)$ could be disgonalised. (Note that through $J_{10} = \frac{1}{2}\sigma_1$ is not diagonal, it can be diagonalised). This completes the classification of and choice of representative elements from the rank 2 orbits of **SO (3,1)**.

Now we consider the rank 4 orbits consisting of $J(\xi)$ with $\mathcal{C}_2(\xi) \neq 0$. Here we can immediately use the results of the Jordan canonical forms : since by eqn. (3.20) det $J(\xi)$ is nonzero, $J(\xi)$ can be diagonalised. Therefore on the orbit of $J(\xi)$ there certainly are elements which in the spinor representation are complex multiples of σ_3. One can easily see then that on a given orbit there is a unique element of the form $uJ_{12}+vJ_{03}$ with $u > 0$ and $v \neq 0$. The values of u and v are unambiguously determined by $\mathcal{C}_1(\xi)$ and $\mathcal{C}_2(\xi)$.

Putting together the two sets of results for both ranks. We get the following table :

SO (3,1) Orbit structure :

Rank (ξ)	Orbit	Parameter ranges	Invariant $\mathcal{C}_1(\xi)$	Invariant $\mathcal{C}_2(\xi)$	Representative Point	Invariant vectors
2	$\vartheta_{3,1}(ts;u)$	$u > 0$	u^2	0	$u\,J_{12}$	e_0, e_3
2	$\vartheta_{3,1}(ls)$	—	0	0	$J_{12} + J_{10}$	e_0+e_2, e_3
2	$\vartheta_{3,1}(ss;v)$	$v > 0$	$-v^2$	0	$v\,J_{10}$	e_2, e_3
4	$\vartheta_{3,1}(u,v)$	$u>0, v \neq 0$	$u^2 - v^2$	uv	$u\,J_{12} + uJ_{03}$	—

It is clear that by a combination of geometrical arguments and matrix-theoretical arguments, the complete results for **SO (3,1)** emerge relatively easily. It is also clear that in all cases, the values of $\mathcal{C}_1(\xi)$ and $\mathcal{C}_2(\xi)$ determine the orbit to which $J(\xi)$ belongs; this contrasts with the situation in **SO (2,1)**.

SO (4,1)

Now we turn to the third and last of the Lorentz type groups to be studied. As in the case of SO (5), here too since the total dimension of the space is five which is odd, every generator $J(\xi)$ has an associated matrix (ξ_{AB}) which has at least one nontrivial null vector. However whereas with **SO (4)** such a result essentially reduced the problen to **SO (4)** (and in suitable circumstances further down to **SO (5)**), in the present case we have a grater variety of possible configurations to consider, due to the changed metric.

Let us begin by listing those expressions for SO (4,1) which are similar to corresponding ones for SO (5) :

$$J(\xi) = \tfrac{1}{2}\, \xi^{AB}\, J_{AB}; \qquad \text{(a)}$$

$$\mathcal{C}_1(\xi) = \tfrac{1}{2}\, \xi^{AB}\, \xi_{AB}; \qquad \text{(b)}$$

$$\zeta_A = \tfrac{1}{8}\, \epsilon_{ABCDE}\, \xi^{BC}\, \xi^{DE}; \qquad \text{(c)}$$

$$\mathcal{C}_2(\xi) = \zeta^A\, \zeta_A. \qquad \text{(d)} \quad (3.21)$$

These are to be supplemented by the identity

$$\xi_{AB}\, \zeta^B = 0. \qquad \ldots(3.22)$$

Depending on whether the rank of (ξ_{AB}) is 2 or 4, we have three or one independent invariant vectors. In the former case, a choice can be made such that they are mutually orthogonal, and if nonlightlike are normalised to $\pm$ 1. We will find that $\zeta^A = 0$ when rank (ξ_{AB}) is 2; and that when rank $(\xi_{AB}) = 4$, ζ^A is the sole nonvanishing invariant vector.

We classify first the rank 2 orbits. The three independent and mutually orthogonal invariant vectors could in principle be of any one of the ten types (listed in dictionary order) *ttt, ttl, tts*; *tll, tls*; *tss*; *lll, lls*; *lss*; *sss*. However, in a space with signature $-++++$, the configurations *tt, tl, ll* cannot occur; i.e., we cannot find two mutually orthogonal time-like vectors etc. Therefore the only possible configurations for the three invariant vectors are three in numbers *tss, lss, sss*. In every case, there are two mutually orthogonl (normalised) space-like vectors. We may conclude: if an element $J(\xi) \in$ **SO (4.1)** with rank 2 is given, there definitely are elements on its orbit which leave e_3 and e_4 invariant. Such elements belong to the **SO (2,1)** generating the SO (2,1) operat ing on indices 012, and so are linear combinations of J_{ab}. Analysis of the rank 2 orbits is thus related to the **SO (2,1)** problem, so there are precisely three mutually exclusive possibilities characterising the orbit of $J(\xi)$: (i) there is an element leaving e_0 invariant, realising the situation *tss*; (ii) there is an element leaving $e_0 + e_2$ invariant, corresponding to *lss*. (iii) there is an element leaving e_2 invariant, corresponding to *lss*. In case (i) there is a unique element uJ_{12}, $u > 0$, on the orbit of $J(\xi)$; the possibility $u < 0$ can be changed to $u > 0$ by a rotation in the 1-4 plane. In case (ii) the element $J_{12} + J_{10}$ is on the orbit; again the element $-(J_{12} + J_{10})$ goes into $J_{12} + J_{10}$ by a suitable 1-4 rotation. Lastly in case (iii) a unique element vJ_{10}, $v > 0$, lies on the orbit. This completes the catalogue of rank 2 orbits. In all cases the vanishing of ζ^A is obvious.

The discussion of the rank 4 orbits brings in a group not encounted in the treatment so far. If a $J(\xi) \in$ **SO (4,1)** of rank 4 be given, there is just one vector which it leaves invariant, which is of type *t*, *l* or *s*. In the *l* and *s* cases, the problem reduces to the subgroup SO (4) or SO (3,1) respectively. However the *t* case involves an E(3) subgroup of SO (4,1), which is an inhomogeneous real orthogonal group. We dispose of the *t* and *s* cases first, then take up the *l* case.

In case a rank 4 orbit in **SO (4,1)** is of type t, there definitely are elements on it which leave e_0, and no other vector, invariant. Such elements belong to **SO (4)** generating the SO (4) group acting on the "space" dimensions 1, ..., 4; and within **SO (4)** they must be of rank 4. One can then check that out of them, a unique representative element $u\,J_{12} + u' J_{43}$ with $u \geqslant |\,u'\,| > 0$ can be picked and can serve as a representative element for the **SO (4,1)** orbit itself. At the representative point the only nonzero component of ζ_A is $\zeta_0 = -\,uu'$, so ζ_A is in the direction $\pm\, e_0$. If the rank 4 orbit is of type s, then e_0, **SO (4)** and SO (4) are replaced by e_4, **SO (3,1)** and SO (3,1) acting on 0, ..., 3 respectively. So elements $uJ_{12} + vJ_{03}$ with $u > 0$, $v \neq 0$ exist on the orbit. But since the freedom of rotations is the 3-4 plane in available in SO (4,1), we can find a unique point on the orbit by requiring both $u > 0$ and $v > 0$, Now, the only nonzero component of ζ_A is $\zeta_4 = uv$, which confirms our expectations.

The last case to be examined in **SO (4,1)** is a rank 4 orbit of type 1. By suitable SO (4,1) transformations, starting with any element on such an orbit, we can pass to those elements for which the (single) invariant vector is e_0+e_4. We must now determine the general form of such elements $J\,(\xi)$, using the property that the matrix (ξ_{AB}) is of rank 4 and annihilates only $z^A = (1, 0, 0, 0, 1)$. Among the equations $\zeta_{AB}\, z^B = 0$ there are four independent ones :

$$\xi_{04} = 0$$

$$\xi_{j4} = -\,\xi_{j0},\quad j = 1, 2, 3. \qquad \text{...(3.23)}$$

So we can write

$$J\,(\xi) = \tfrac{1}{2}\,\xi^{AB}\,J_{AB}$$

$$= \tfrac{1}{2}\,\xi_{jk}\,J_{jk} + \xi_{j0}\,(J_{0j} + J_{4j}). \qquad \text{...(3.24)}$$

Here, as in the treatment of SO (3) within SO (3,1), the indices j, k, ... range over 1,2, 3. The generators **J**, **P** $\in$ **SO (4,1)** defined as

$$J_j = \tfrac{1}{2}\,\epsilon_{jkl}\,J_{kl},$$

$$P_j = J_{0j} + J_{4j} \qquad \text{...(3.25)}$$

obey the brackets relations of E (3), the Lie algebra of the Euclidean group in three dimensions :

$$[J_j, J_k] = \epsilon_{jkl}\,J_l$$

$$[J_j,\ P_k] = \epsilon_{jkl}\ P_l,$$

$$[P_j, P_k] = 0. \qquad \text{...(3.26)}$$

This is expected, as the stability group of a light like vector in 1 + 4 space is E (3). Now we must search for the necessary and sufficient conditions on (ξ_{AB}) apart from (3.23) which ensure that $e_0 + e_4$ is the 'only' vector invariant under $J\,(\xi)$. Let us write

the $J(\xi)$ in eqn. (3.24) as

$$J(\xi) = \xi.\mathbf{J} + \alpha.\mathrm{P},$$

$$\xi_l = \tfrac{1}{2}\, \epsilon_{jkl}\, \xi_{kl},$$

$$\alpha_j = \xi_{j0}. \qquad \text{...(3.27)}$$

The condition that this (ξ_{AB}) annihilate a general z^A is

$$\alpha, \mathrm{z} = 0,$$

$$\xi_\wedge \mathrm{z} + (z^0 - z^4)\, \alpha = 0. \qquad \text{...(3.28)}$$

It is an easy consequence of these equations that

$$\xi.\alpha\, \mathrm{z} = 0,$$

$$\xi.\alpha\, (z^0 - z^4) = 0. \qquad \text{...(3.29)}$$

Now from eqn. (3.28) we see that if either ξ or α were to vanish, the desired conclusion on z^A, namely $\mathrm{z} = 0$ and $z^0 = z^4$, would not follow; hence we must insist that both ξ and α be nonzero. If $\xi.\alpha = 0$, then from eqn. (3.29) we see that the desired conclusion on z^A does follow. On the other hand, if $\xi.\alpha \neq 0$ with neither ξ nor α vanishing individually, we can find a solution to eqn. (3.28) setting $z^0 = z^4$ and z proportional to ξ. Therfore the necessary and sufficient condition we are seeking on $J(\xi) \in$ E (3) is $\xi\, \alpha \neq 0$.

We have in this way found the general form of elements on the given 1-type orbit, where the invariant vector is $e_0 + e_4$. These elements can be denoted by pairs (ξ, α). Now we must exploit the adjoint action by E (3) to try and put such a pair into some simple and natural from. The adjoint action by the SO (3) part of E (3) is given by

$$(\xi, \alpha) \rightarrow (R\,\xi, R\,\alpha), \qquad \text{...(3.30)}$$

where $R \in$ SO (3). On the other hand the translation by an amount b acts in the adjoint representation in this way :

$$(\xi, \alpha) \rightarrow (\xi, \alpha + \mathbf{b} \wedge \xi). \qquad \text{...(3.31)}$$

Under these changes (3.30, 31), $\xi.\,\alpha$ is invariant. Now given a pair (ξ, α) with $\xi.\alpha \neq 0$, we can first use the freedom (3.30) to put ξ into the form

$$\alpha \rightarrow (0, 0, u),\ u > 0. \qquad \text{...(3.32)}$$

After this, the translational freedom (3.31) can be used to reduce the first two components of α to zero :

$$\alpha \rightarrow (0, 0, \alpha)\ \alpha \neq 0. \qquad \text{...(3.33)}$$

Having achieved this, we see that $J(\xi)$ has been carried by adjoint action using E (3)

alone to the form

$$J(\xi) \to u\, J_{12} + \alpha\, P_3$$
$$= u\, J_{12} + \alpha\, (J_{03} + J_{43}). \qquad \ldots(3.34)$$

At this point we have to ask if the further freedom of transformation remaining in SO (4.1) after all of E (3) has been used up will help simplify the expression (3.34) still further. It turns out that this is possible: the generator $J_{04} \in$ **SO (4.1)** has the effect of changing the scale of P_j:

$$[J_{01}, J_j] = 0,$$
$$[J_{04}, P_j] = -\, P_j. \qquad \ldots(3.35)$$

Therefore by using finite transformations generated by J_{04}, we can reduce the parameter α in (3.44) to $\epsilon = \pm 1$, depending on the sign of $\xi.\ \alpha$ in the original pair (ξ, α).

Thus we have succeeded in showing that on a rank 4 orbit in **SO (4,1)** of type 1, there is a unique representative element $uJ_{12} + \epsilon\,(J_{03} + J_{43})$ for some $u > 0$ and some $\epsilon = \pm 1$. We can then calculate the five-vector ζ^A at this representative point and find $\zeta^A = \epsilon\,(1, 0, 0, 0, 1)$, as expected.

The complete table of results for **SO (4,1)** is:

SO (4,1) orbit structure:

Rank (ξ)	Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Invariant $\mathscr{C}_2(\xi)$	Representative Point	Invariant vectors
2	$\vartheta_{t,1}\,(tss;\,u)$	$u>0$	u^2	0	uJ_{12}	$e_0, e_3\ e_4$
2	$\vartheta_{4,1}\,(lss)$	–	0	0	$J_{12}+J_{10}$	$e_0+e_2, e_3\ e_4$
2	$\vartheta_{4,1}\,(sss;\,v)$	$v>0$	$-v^2$	0	vJ_{10}	e_2, e_3, e_4
4	$\vartheta_{4,1}\,(t;u,u')$	$u \geqslant \lvert u' \rvert > 0$	$u^2+u'^2$	$-u^2u'^2$	$uJ_{12}+u'J_{43}$	e^0
4	$\vartheta_{4,1}\,(l;\,u,t)$	$u>0;\ \epsilon=\pm 1$	u^2	0	$uJ_{12}+\epsilon(J_{03}+J_{43})$	e_0+e_4
4	$\vartheta_{4,1}\,(s;u,v)$	$u>0;\ v>0$	u^2-v^2	u^2v^2	$uJ_{12}+vJ_{03}$	e_4

It is obvious from this table that the invariants $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ no longer suffice to fix the orbit to which $J(\xi)$ belongs.

4. Orbits in the Lie Algebras SO (n,2), $n = 2$ and 3

The two "de Sitter" type groups SO (n,2) are the last ones we analyse in this paper. The relevant dimensions are numbered 0, 1, 2, 3, 5 with signature $-+++-$. For SO (2,2) we have indices μ, ν, ... and the dimension 3 is omitted. The generators $J_{\mu\nu}$ obey

$$[J_{\mu\nu}, J_{\rho\sigma}] = g_{\mu\rho}\, J_{\nu\sigma} - g_{\nu\rho}\, J_{\mu\sigma} + g_{\mu\sigma}\, J_{\rho\nu} - g_{\nu\sigma}\, J_{\rho\mu}. \qquad \ldots(4.1)$$

For SO (3,2) we replace $\mu\nu\rho\sigma$ by $ABCD$ going over the full range. For these two groups the antisymmetric symbols are normalised by

$$\epsilon_{01235} = \epsilon_{0125} = 1 \qquad \text{...(4.2)}$$

SO (2,2)

Just as with **SO (4)** in Section 2 where the orbit classification was simplified because of the local decomposition SO (4) $\simeq$ SO (3) $\otimes$ SO (3), here we can use the local decomposition SO (2,2) $\simeq$ SO (2,1) $\otimes$ SO (2,1) and the results pertaining to **SO (2,1)**. However since **SO (2,1)** has a much richer orbit structure than SO (3), there being the various types tu, $l\epsilon$ and sv, many more possibilities arise with **SO (2,2)** than arose with SO (4). More over we must remember that the t, l and s classification of orbits within each **SO (2,1)** factor in SO (2,2) has no such geometrical interpretation in four dimensions[18].

Adapting eqns. (2.12, 14) to the present situation we define :

$$J_a = \tfrac{1}{2}\epsilon_{abc} J^{bc}, \quad K_a = J_{5a};$$

$$M_a = \tfrac{1}{2}(J_a + K_a), \quad N_a = \tfrac{1}{2}(J_a - K_a); \; a, b = 0, 1, 2. \qquad \text{...(4.3)}$$

Then eqns. (4.1) can be expressed in two ways:

$$[J_a, J_b] = [K_a, K_b] = \epsilon^{c}_{ab} J_c,$$

$$[J_a, K_b] = \epsilon^{c}_{ab} K_c; \qquad \text{(a)}$$

$$[M_a, M_b] = \epsilon^{c}_{ab} M_c; \quad [N_a, N_b] = \epsilon^{c}_{ab} N_c;$$

$$[M_a, N_b] = 0. \qquad \text{(b)...(4.4)}$$

These show the decomposition **SO (2,2)** = **SO (2,1)** $\oplus$ **SO (2,1)**. For the components $\xi^{\mu\nu}$ of a general element $J(\xi) \in$ SO (2,2) we define :

$$\xi^a = -\tfrac{1}{2}\epsilon^{abc}\xi_{bc}, \; \eta^a = \xi^{5a};$$

$$\alpha^a = \xi^a + \eta^a, \; \beta^a = \xi^a - \eta^a. \qquad \text{...(4.5)}$$

Then $J(\xi)$ and the two algebraic invariants are :

$$J(\xi) = \tfrac{1}{2}\xi^{\mu\nu} J_{\mu\nu} = \xi^a J_a + \eta^a K_a = \alpha^a M_a + \beta^a N_a;$$

$$\mathscr{C}_1(\xi) = \tfrac{1}{2}\xi^{\mu\nu}\xi_{\mu\nu} = -\xi^a\xi_a - \eta^a\eta_a = -\tfrac{1}{2}(\alpha^a\alpha_a + \beta^a\beta_a);$$

$$\mathscr{C}_2(\xi) = \tfrac{1}{8}\epsilon_{\mu\nu\rho\sigma}\xi^{\mu\nu}\xi^{\rho\sigma} = \xi^a\eta_a = \tfrac{1}{4}(\alpha^a\alpha_a - \beta^a\beta_a). \qquad \text{...(4.6)}$$

The invariant $\mathscr{C}_2(\xi)$ determines the rank of $(\xi_{\mu\nu})$ since

$$\Delta(\xi) = \det(\xi_{\mu\nu}) = (\xi^a\eta_a)^2 = (\mathscr{C}_2(\xi))^2. \qquad \text{...(4.7)}$$

Therefore $\mathscr{C}_2(\xi) = 0\ (\neq 0)$ corresponds to rank $(\xi_{\mu\nu}) = 2\ (4)$.

The adjoint action of SO (2,2) on $\xi^{\mu\nu}$ amounts to the following: we subject α^a and β^a to independent SO (2,1) transformations as 3-vectors, corresponding to the two factors in the product SO (2,2) $\simeq$ SO (2,1) $\otimes$ SO (2,1). Therefore each orbit in SO (2,2) is the Cartesian product of two SO (2,1) orbits, with α^a lying on the first factor and β^a on the second. In looking for nontrivial **SO (2,2)** orbits, we can allow at most one factor in the product to be trivial. Thus to begin with, the distinct SO (**2,2**) orbits can be listed in this way: $(0;\ tu_2)$, $(0;\ l\,\epsilon_2)$, $(0;\ sv_2)$; $(tu_1;\ 0), \ldots (tu_1;\ sv_2);\ \ldots;(sv_1;0)\ldots,(sv_1;\ sv_2)$. Here u_1, u_2 $\neq 0$; $\epsilon_1, \epsilon_2 + \pm 1$; $v_1, v_2 > 0$; and there are fifteen combinations. On any one of these **SO (2,2)** orbits, a unique representative element is obtained as the sum of representative elements from each factor. As examples we have:

$$(tu_1;\ 0) \rightarrow u_1\, M_0 = \tfrac{1}{2}\, u_1\, (J_{12} + J_{50});$$

$$(l\epsilon_1;\ sv_2) \rightarrow \epsilon_1\,(M_0 + M_2) + v_2 N_2$$

$$= \tfrac{1}{2}\, v_2\,(J_{10} - J_{52}) + \tfrac{1}{2}\,\epsilon_1\,(J_{10} + J_{12} + J_{50} + J_{52}). \qquad \ldots(4.8)$$

In each case, the values of $\xi_{\mu\nu}$ at the representative element can be read off, and then $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ for the entire orbit calculated. Towards classifying SO (2,2) orbits according to their ranks in the four-dimensional sense, we give in a table the value of $4\,\xi^a\eta_a = 4\,\mathscr{C}_2(\xi)$ in each cartesian product. The rows (columns) are labelled by the first (second) factor in the product.

Values of $4\,\mathscr{C}_2(\xi)$:

	0	tu_2	$l\epsilon_2$	sv_2
0	—	u_2^2	0	$-v_2^2$
tu_1	$-u_1^2$	$u_2^2 - u_1^2$	$-u_1^2$	$-u_1^2 - v_2^2$
$l\epsilon_1$	0	u_2^2	0	$-v_2^2$
sv_1	v_1^2	$v_1^2 + u_2^2$	v_1^2	$v_1^2 - v_2^2$

Since the total number of orbit types for **SO (2,2)** (and later also for **SO (3,2)**)) is quite large, we present the pattern of rank 2 orbits separately from that of rank 4 orbits. As a first step we read off from the above table all those cases when $\mathscr{C}_2(\xi) = 0$, corresponding to $J(\xi)$ being of rank 2, and also in each case write down a representative element, as was donn in the examples of (4.8) (the preliminary expressions in terms of M_a and N_a are omitted):

$$(tu_1;\, tu_1) \rightarrow u_1\, J_{12} \tag{i}$$

$$(l\epsilon_1;\, l\epsilon_1) \rightarrow \epsilon_1\, (J_{10} + J_{12}) \tag{ii}$$

$$(sv_1;\, sv_1) \rightarrow v_1\, J_{10} \tag{iii}$$

$$(l\epsilon_1;\, 0) \rightarrow \tfrac{1}{2}\,\epsilon_1\, (J_{10} + J_{12} + J_{50} + J_{52}) \tag{iv}$$

$$(0;\, l\epsilon_2) \rightarrow \tfrac{1}{2}\,\epsilon_2\, (J_{10} + J_{12} - J_{50} - J_{52}) \tag{v}$$

$$(l\epsilon_1;\, l,\, -\epsilon_1) \rightarrow \epsilon_1\, (J_{50} + J_{52}) \tag{vi}$$

$$(tu_1;\, t,\, -u_1) \rightarrow u_1\, J_{50}. \qquad \text{(vii)} \quad (4.9)$$

The reason for listing these seven cases in this particular order will become clear soon. Now in each of these cases we know, since $\mathscr{C}_2(\xi) = 0$, that there are two independent (mutually orthogonal) vectors in four dimensions which are annihilated by the matrix $(\xi_{\mu\nu})$ at the representative point. These pairs of vectors are easily calculated and for the seven situations listed in (4.9) they are, in the same order: e_0, e_5; $e_0 + e_2, e_5$; e_2, e_5; $e_0 + e_2, e_1 + e_5$; $e_0 + e_2, e_1 - e_5$; $e_0 + e_2, e_1$; e_1, e_2. We see that in case (i) the representative generator $J(\xi)$ leaves invariant two (mutually orthogonal) unit time like vectors, so it is a realisation of the possibility tt for the invariant vectors 'in the four-dimensional sense', i. e. in the 0125 space on which SO (2,2) acts. Similarly case (ii) corresponds to the configuration tl; and the remaining ones to ts, ll, ll, ls and ss in that order. (Now we see that the sequence in (4.9) corresponds to dictionary order in the symbols t, l, s interpreted in the four dimensional sense). The appearance of two inequivalent ll configurations, namely $e_0 + e_2, e_1 + e_5$ and $e_0 + e_2, e_1 - e_5$ is to be noted: they cannot be transformed into one another by any SO (2,2) transformation[42]. In all the other cases, namely tt, tl, ts, ls and ss, any configuration of the concerned type can be transformed via SO (2,2) into the given configuration. The complete list of rank 2 orbits in SO (2,2) can now be tabulated. We drop the subscripts 1,2 on the parameters in (4.9), and in cases (iv) and (v) we use the scaling freedom provided by the generators M_1 and N_1 respectively to replace $\frac{1}{2}\epsilon$ by ϵ. Thus we arrive at the following table, where in the first column the case number taken from (4.9) is given :

Rank 2 orbit structure in **SO (2,2)** : $\mathscr{C}_2(\xi) = 0$:

Case In(4.9)	Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Representative Point	Invariant vectors
(i)	$\vartheta_{2,2}(tt;\, u)$	$u \neq 0$	u^2	uJ_{12}	e_0, e_5
(ii)	$\vartheta_{22}(tl;\, \epsilon)$	$\epsilon = \pm 1$	0	$\epsilon\,(J_{10} + J_{12})$	$e_0 + e_2,\, e_5$
(iii)	$\vartheta_{2\cdot 2}(ts;\, v)$	$v > 0$	$-v^2$	vJ_{10}	$e_2,\, e_5$

(*table continued on p.* 113)

Case	Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Representative Point	Invariant vectors
(iv)	$\vartheta_{2,2}(ll;\epsilon)$	$\epsilon=\pm 1$	0	$\epsilon(J_{10}+J_{12}+J_{50}+J_{52})$	e_0+e_2, e_1+e_5
(v)	$\vartheta'_{22}(ll;\epsilon)$	$\epsilon=\pm 1$	0	$\epsilon(J_{10}+J_{12}-J_{50}-J_{52})$	e_0+e_2, e_1-e_5
(vi)	$\vartheta_{22}(ls;\epsilon)$	$\epsilon=\pm 1$	0	$\epsilon(J_{50}+J_{50})$	e_0+e_2, e_1
(vii)	$\vartheta_{22}(ss;u)$	$u\neq 0$	u^2	uJ_{50}	$e_1, e_2.$

One can see already at the level of rank 2 orbits the extent to which the values of the algebraic invariants $\mathscr{C}_1(\xi)$ and $\mathscr{C}_2(\xi)$ fail to fix the orbit to which $J(\xi)$ belongs. If one also compares the patterns of rank 2 orbits in **SO (4)**, **SO (3,1)** and **SO (2,2)** with one another, all of the corresponding groups being defined on a four-dimensional space, one can see a gradual increase in complexity as the metric changes from Euclidean to Lorentzian to de Sitter.

When we turn next to cataloguing the rank 4 orbits in **SO (2,2)** their variety is again vastly greater than with either **SO(4)** or **SO (3,1)**. Recall that in the latter cases, these orbits can be compactly denoted as $\vartheta_4(u, u')$, $\vartheta_{3,1}(u, v)$ respectively, with uniformly valid expressions for the invariants and representative elements. With **SO(2,2)** the situation will turn out to be very different. One of the aspeets requiring specific attention will be that of finding suitable symbols for the various distinct families of orbit since the labels t,l,s are no longer available, there being no nullvectors for a $(\xi_{\mu\nu})$ of rank 4.

Going through the table of values of $4\mathscr{C}_2(\xi)$ row by row, we find in the first instance twelve types of **SO (2,2)** orbits over which $\mathscr{C}_2(\xi)$ does not vanish. As in (4.9), we list these cases now, in the sequence in which they occur in the $\mathscr{C}_2(\xi)$ table, giving in each case the corresponding representative element as a liaear combination of $J_{\mu\nu}$:

$$(0{:}\, tu_2) \rightarrow \tfrac{1}{2}u_2(J_{12}-J_{50}) \tag{i}$$

$$(0;\, sv_2) \rightarrow \tfrac{1}{2}v_2(J_{10}-J_{52}) \tag{ii}$$

$$(tu_1;\, 0) \rightarrow \tfrac{1}{2}u_1(J_{12}+J_{50}) \tag{iii}$$

$$(tu_1{:}\, tu_2),\ u_1\neq\pm u_2 \rightarrow \tfrac{1}{2}(u_1+u_2)J_{12}+\tfrac{1}{2}(u_1-u_2)J_{50} \tag{iv}$$

$$(tu_1;\, l\epsilon_2) \rightarrow \tfrac{1}{2}u_1(J_{12}+J_{50})+\tfrac{1}{2}\epsilon_2(J_{10}+J_{12}-J_{50}-J_{52}) \tag{v}$$

$$(tu_1;\, sv_2) \rightarrow \tfrac{1}{2}u_1(J_{12}+J_{50})+\tfrac{1}{2}v_2(J_{10}-J_{50}-J_{52}) \tag{vi}$$

$$(l\epsilon_1;\, tu_2) \rightarrow \tfrac{1}{2}u_2(J_{12}-J_{50})+\tfrac{1}{2}\epsilon_1(J_{10}+J_{12}+J_{50}+J_{52}) \tag{vii}$$

$$(l\epsilon_1;\, sv_2) \rightarrow \tfrac{1}{2}v_2(J_{10}-J_{52})+\tfrac{1}{2}\epsilon_1(J_{10}+J_{12}+J_{50}+J_{52}) \tag{viii}$$

$$(sv_1;\,0) \rightarrow \tfrac{1}{2}v_1\,(J_{10} + J_{52}) \tag{ix}$$

$$(sv_1;\,tu_2) \rightarrow \tfrac{1}{2}\,v_1\,(J_{10} + J_{52}) + \tfrac{1}{2}\,u_2\,(J_{12} - J_{50}) \tag{x}$$

$$(sv_1\,;l\epsilon_2) \rightarrow \tfrac{1}{2}\,v_1\,(J_{10} + J_{52}) + \tfrac{1}{2}\,\epsilon_2\,(J_{10} + J_{12} - J_{50} - J_{52}) \tag{xi}$$

$$(sv_1;\,sv_2),\; v_1 \neq v_2 \rightarrow \tfrac{1}{2}\,(v_1 + v_2)\,J_{10} + \tfrac{1}{2}\,(v_1 - v_2)\,J_{52}. \qquad \text{(xii)} \ldots (4.10)$$

(We remind ourselves that every $u_i \neq 0$, every $v_i > 0$ and every $\epsilon_i = \pm 1$). Naturally none of these representative elements has any invariant vector. Now our task is to combine sets of these families of orbits judiciously to form larger coherent families, based on the forms of the representative elements. One can see for example that the representative elements in cases (i), (iii) and (iv) combine neatly into the two-parameter family $uJ_{12}+u'J_{50}$ subject to the restrictions $u,\, u' \neq 0$; here we have identified $\frac{1}{2}\,(u_1 \pm u_2)$ with $u,\, u'$ respectively. Let us call this family of orbits as famiiy A, so an individual member of it is written $\vartheta_{2,2}\,(A;\, u,\, u')$. Similarly, cases (ii), (ix) and (xii) in (4.10) have representative elements combining neatly into the expression $vJ_{10} + v'J_{52}$ subject to $v \geqslant |\, v'\,| > 0$. These restrictions on v,v' result from identifying them with $\frac{1}{2}(v_1 \pm v_2)$ respectively. This family of orbits will be labelled by the letter B, leading to $\vartheta_{2,2}\,(B;\, v,\, v')$. Thus six of the twelve cases listed in (4.10) are taken care of, leaving six more to be handled, namely (v), (vi), (vii), (viii), (x) and (xi).

Now these six cases split naturally into three pairs: (v) and (vii), (vi) and (x), (viii) and (xi). Within each pair, the relationship is that the representative elements get interchanged by the reversal of the sign of the dimension 5 (and accompanying relabelling of parameters). This discrete operation is an outer automorphism on SO (2,2), not an element in the identity component of SO (2,2) and it amounts to interchanging the SO (2,1) factors in the (local) product SO (2,2) $\simeq$ SO (2,1) $\otimes$ SO (2,1). Thus within SO (2,2), it is the interchange $M_a \leftrightarrow N_a$. Taking up fiirst the pair (vi), (x) in (4.10): we replace $u_{1,2} \rightarrow 2u$, $v_{1,2} \rightarrow 2v$, and introduce a sign parameter $\epsilon' = \pm 1$ to distinguish cases (vi) and (x) respectively. Thus we get a combined expression $u\,(J_{12} + \epsilon' J_{50}) + v\,(J_{10} - \epsilon' J_{52})$ for the representative element on an orbit of family C, with the parameter conditions $u \neq 0,\, v > 0,\, \epsilon' = \pm 1$. For the pair (v), (vii): we use the scaling freedom via transformations generated by $M_1,\, N_1$ to alter $\frac{1}{2}\,\epsilon_{1,2}$ to $\epsilon_{1,2}$; then with the changes $\epsilon_{1,2} \rightarrow \epsilon$, $u_{1,2} \rightarrow 2u$, and introduction of $\epsilon' = \pm 1$ to distinguish between cases (v) and (vii), we get the representative element $u(J_{12} + \epsilon'\, J_{50}) + \epsilon\,(J_{10} + J_{12} - \epsilon'\, J_{50} - \epsilon'\, J_{52})$ on an orbit of family D, subject to $u \neq 0,\, \epsilon,\, \epsilon' = \pm 1$. For the last pair (viii), (xi) in (4.10): by simillar steps we get an element $v\,(J_{10} - \epsilon'\, J_{52}) + \epsilon\,(J_{10} + J_{12} + \epsilon'\, J_{50} + \epsilon'\, J_{52})$, $v > 0,\, \epsilon,\, \epsilon' = \pm 1$, representing an orbit in the family E.

By this reorganisation of the entries in (4.10), the rank 4 orbits in SO (2,2) fall into five major families. Calculation of $\mathscr{C}_1\,(\xi)$, $\mathscr{C}_2\,(\xi)$ in each case is straightforward, and the final results are presented in the table on p 115.

Rank 4 orbit structure in SO (2,2):

Cases in (4.10)	Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Invariant $\mathscr{C}_2(\xi)$	Representative point
(i), (iii), (iv)	$\vartheta_{2,2}(A; u, u')$	$u, u' \neq 0$	$u^2 + u'^2$	$-uu'$	$uJ_{12} + u' J_{50}$
(ii), (ix), (xii)	$\vartheta_{2,2}(B; v, v')$	$v \geq \lvert v' \rvert > 0$	$-v^2 - v'^2$	vv'	$vJ_{10} + v' J_{52}$
(vi), (x)	$\vartheta_{2,2}(C, \epsilon': u, v)$	$u \neq 0, v > 0,$ $\epsilon' = \pm 1$	$2(u^2 - v^2)$	$-\epsilon'(u^2 + v^2)$	$u(J_{12} + \epsilon' J_{50}) + v(J_{10} - \epsilon' J_{52})$
(v), (vii)	$\vartheta_{2,2}(D, \epsilon'; u, \epsilon)$	$u \neq 0, \epsilon' = \pm 1,$ $\epsilon = \pm 1$	$2u^2$	$-\epsilon' u^2$	$u(J_{12} + \epsilon' J_{50})$ $+ \epsilon(J_{10} + J_{12} - \epsilon' J_{60} + \epsilon' J_{52})$
(viii), (xi)	$\vartheta_{2,2}(E, \epsilon'; v, \epsilon)$	$v > 0, \epsilon' = \pm 1,$ $\epsilon = \pm 1$	$-2v^2$	$-\epsilon' v^2$	$v(J_{10} - \epsilon' J_{52})$ $+ \epsilon(J_{10} + J_{12} + \epsilon' J_{50} + \epsilon' J_{52})$

The complete picture of all **SO (2,2)** orbits is obtained by combining the two tables refering to rank 2 and rank 4 orbits respectively. It may be useful to mention here that in any practical application of these results, the identification of the orbit to which a given $J(\xi) \in$ **SO (2,2)** belongs is best done by splitting it into its M_a and N_a components, checking with the cases in (4.9) or (4.10) as appropriate, and then assingning it to the correct $\vartheta_{2,2}(\ldots)$. The algebraic invariants $\mathscr{C}_1(\xi)$, $\mathscr{C}_2(\xi)$ do not by themselves determine the orbit.

SO (3,2)

For this group all of eqns. (3.21) giving expressions for $J(\xi)$, $\mathscr{C}_1(\xi)\,\zeta_A$ and $\mathscr{C}_2(\xi)$ in the case of **SO (4.1)** can be taken over as they stand, with the understanding that A, B, ... now go aver 0 1 2 3 5 with signature $-+++-$. Moreover eqn. (3.22) also remains valid. The rank of (ξ_{AB}) is either 2 or 4, leading to the existence of 3 or 1 independent invariant vectors. The former corresponds to rank 2 orbits which we take up first.

As with **SO (4,1)**, we first list all ten conceivable configurations of three (mutually orthogonal) invariant vectors in dictionary order: *ttt*, *ttl*, *tts*; *tll*, *tls*; *tss*; *lll*, *lls*; *lss*; *sss*. While with the signature of a $4+1$ space only three of these actually exist, in a $3+2$ space six configurations survive and these are: *tts*, *tls*, *tss*; *lls*, *lss*, *sss*. These have been split into two sets of three configurations each because, as we shall soon see, the first set can be handled at the **SO (2,1)** level, while the second set in reducible to a problem within **SO (2,2)**.

Let $J(\xi) \in$ **SO (3,2)** be of rank 2, and let the null vectors of (ξ_{AB}) be invariantly characterised as being of one of the types *tts*, *tls* or *tss*. In every case we have one time like and one space like vector, mutually orthogonnl and normalised, included in the triad. One can therefore always pass via suitable SO(3,2) transformations to element(s) on the orbit of $J(\xi)$ which leave e_5 and e_3 invariant. Such element(s) then belong to the **SO (2,1)** algebra associated with the dimensions 0, 1, 2. The further separation into three mutually exclusive possibilities corresponds to whether the third invariant vector is of type *t*, *l* or *s* in the 0 1 2 subspace. Therefore the orbit of $J(\xi)$ definitely contains element(s) of one of the following three types: uJ_{12} with $u \neq 0$ or $\epsilon\,(J_{12} + J_{10})$ with $\epsilon = \pm 1$ or vJ_{10} with $v > 0$. In the first two cases, the sign of u or ϵ can be arranged to be positive, if necessary by making a suitable rotation in the 1-3 plane. This settles the quesition of finding suitable representative elements for rank 2 orbits of types *tts*, *tls* and *tss*.

Turning to the three remaining rank 2 orbits of types *lls*, *lss* and *sss*, we see that in every case there is at least one unit space like vector in the invariant triad. On all such orbits there are then elements which leave e_3 and two other vectors invariant. Such elements therefore lie in **SO (2,2)** associated with the dimensions 0 1 2 5, which has been analysed earlier in this Section; further they are of rank 2 within this **SO(2,2)**.

The classification of such **SO** (2,2) orbits shows that with the help of SO (2,2) transformations we can pass to elements which, in addition to e_3, leave invariant one of the following four pairs of vectors, depending on the invariant configuration associated with $J(\xi)$:

$$
\begin{aligned}
&lls \rightarrow e_0 + e_2, e_1 + e_5 \text{ or } e_0 + e_2, e_1 - e_5;\\
&lss \rightarrow e_0 + e_2, e_1;\\
&sss \rightarrow e_1, e_2.
\end{aligned}
\qquad (4.11)
$$

However, while the two possible pairs associated with lls are inequivalent at the SO(2,2) level, they are transformable into one another by a suitable 1-3 rotation within SO(3,2). Therefore, depending on whether the orbit of $J(\xi)$ is of type lls, lss, or sss, there is a 'unique' element on it leaving $e_0 + e_2$, $e_1 + e_5$, e_3, or $e_0 + e_2$, e_1, e_3 or e_1, e_2, e_3 respectively invariant; and the form of this element can be taken from the table of rank 2 orbits in **SO (2,2)**, namely case (iv), (vi) or (vii) respectively in that table. Now this table shows that in cases (iv) and (vi) there is a parameter ϵ which can take values ± 1, and these are distinct possibilities within SO (2,2). Similarly in case (vii) of that table the parameter u can be positive or negative. One must naturally examine whether, in view of the greater freedom of transformation available in SO (3,2), the parameter ϵ could be restricted to $+1$, and u to positive values alone. This however cannot be achieved, since it requires (among other things) swiching the sign of J_{50}.

Taking into account the results of the two preceding paragraphs, we can construct a catalogue of all the rank 2 orbits in **SO (3,2)**. It is easily seen that on all such orbits, $\zeta_A = 0$ identically, so $\mathscr{C}_2(\xi) = 0$ as well. These facts are explicity indicated in the following tables :

Rank 2 orbit structure in **SO (3,2)** : $\zeta_A = \mathscr{C}_2(\xi) = 0$:

Orbit	Parameter range	invariant $\mathscr{C}_1(\xi)$	Representative point	Invariant vectors
$\vartheta_{3,2}(tts; u)$	$u > 0$	u^2	uJ_{12}	e_0, e_3, e_5
$\vartheta_{3,2}(tls)$	—	0	$J_{10} + J_{12}$	$e_0 + e_2, e_3, e_5$
$\vartheta_{3,2}(tss; v)$	$v > 0$	$-v^2$	vJ_{10}	e_2, e_3, e_5
$\vartheta_{3,2}(lls; \epsilon)$	$\epsilon = \pm 1$	0	$\epsilon(J_{10} + J_{12} + J_{50} + J_{52})$	e_0+e_2, e_1+e_5, e_3
$\vartheta_{3,2}(lss; \epsilon)$	$\epsilon = \pm 1$	0	$\epsilon(J_{50} + J_{52})$	e_0+e_2, e_1, e_3
$\vartheta_{3,2}(sss; u)$	$u \neq 0$	u^2	uJ_{50}	e_1, e_2, e_3

Let us now turn to our last topic, the analysis of orbits of rank 4 in **SO (3,2)**. Following what is by now a familiar pattern, these orbits in every case can be related to

some suitable six-dimensional subalgebra in **SO** (3,2). As we would expect, it will happen that for any rank 4 $J(\xi) \in$ **SO** (3,2), ζ_A is nonvanishing and is the sole nullvector of (ξ_{AB}). The possible orbits therefore split initially into three types, corresponding to ζ_A being of type t, l or s. We can pass via suitable SO (3,2) transformations to elements on the orbit of $J(\xi)$ for which the invariant vector ζ_A is proportional to e_5, $e_5 + e_3$ or e_3 respectively. Such elements must then belong to the **SO (3,1)** subalgebra associated with the dimensions 0 1 2 3, an **E (2,1)** subalgebra (as we shall see), or the **SO (2,2)** subalgebra associated with the dimensions 0 1 2 5. In all cases, we have to deal with rank 4 elements in these subalgebras. Except for the **E (2,1)** case, then, we can draw on previously derived results.

The case when ζ_A is of type t is easiest to handle. Then, among the elements on the orbit of $J(\xi)$ which leave e_5 invariant are some of the form $uJ_{12} + vJ_{03}$ for some $u > 0$, $v \neq 0$. This much follows from the nature of rank 4 orbits in **SO (3,1)**. But going beyond this, the freedom to perform rotations in the 0-5 plane shows that we can arrange for v also to be positive. The result is that on any rank 4 orbit in **SO** (3,2) of type t, there is a unique element $uJ_{12} + vJ_{03}$ with both $u, v > 0$.

Next we consider the case when ζ_A is of type l, and ask for the most general $J(\xi)$ for which the only invariant vector is $e_3 + e_5$. Thus (ξ_{AB}) must annihilate only $z^A = (0, 0, 0, 1, 1)$. The equations $\xi_{AB}\, z^B = 0$ give the following conditions:

$$\xi_{35} = 0,$$

$$\xi_{a3} = -\,\xi_{a5}, \quad a = 0,\ 1,\ 2. \qquad \ldots(4.12)$$

This allows ξ_{ab} and ξ_{a3} to be independent. Using a notation patterned after that of section 3 in dealing with **E** (3), we write $J(\xi)$ as

$$J(\xi) = \xi^a\, J_a + \alpha^a\, P_a,$$

$$J_a = \tfrac{1}{2}\, \epsilon_{abc}\, J^{bc},$$

$$P_a = J_{a3} + J_{a5},$$

$$\xi^a = -\,\tfrac{1}{2}\, \epsilon^{abc}\, \xi_{bc}, \quad \alpha^a = \xi^{a3}. \qquad \ldots(4.13)$$

The latin indices here and handled exactly as in the treatment of SO (2,1) in section 3. J_a and P_a span an **E (2,1)** sub-algebra within **SO (3,2)**, i. e. a Poincaré algebra in a $2 + 1$ space, which is the stability group of a light like vector in de Sitter space

$$[J_a,\ J_b] = \epsilon_{ab}^{\ \ c}\, J_c,$$

$$[J_a,\ P_b] = \epsilon_{ab}^{\ \ c}\, P_c,$$

$$[P_a,\ P_b] = 0. \qquad \ldots(4.14)$$

Now we seek necessary and sufficient conditions on ξ^a, α^a to ensure that $e_3 + e_5$ is the only null vector for (ξ_{AB}). The condition that (ξ_{AB}) annihilate a general z^A is :

$$\alpha^a z_a = 0,$$

$$\epsilon_{abc}\, \xi^b z^c - \alpha_a (z^3 - z^5) = 0. \qquad \text{...(4.15)}$$

A consequence of these equations is the pair

$$\alpha^a \xi_a z_b = 0,$$

$$\alpha^b \xi_a (z^3 - z^5) = 0. \qquad \text{...(4.16)}$$

If $\alpha^a \xi_a \neq 0$, that is sufficient to lead to the desired results on z^A, namely, $z^a = 0$ and $z^3 = z^5$. The necessity of this condition is also eassy to prove. Therefore, on a given rank 4 orbit in **SO** (3,2) of type *l*. those elements $J(\xi)$ for which the invariant vector is $e_3 + e_5$ are of the form (4.13) with nonvanishing $\alpha^a \xi_a$, and therefore also with nonvanishing α^a and ξ^a.

In finding a natural representative element on such an orbit, we first use the E(2,1) adjoint action to simplify the pair (ξ^a, α^a) as much as possible, then search for further simplification using elements of SO (3,2) outside E (2,1). The adjoint actions by the homogeneous SO (2,1) part of E (2,1) and by the translations in E (2,1) are :

$$(\xi^a, \alpha^a) \rightarrow (\Lambda^a_b\, \xi^b, \Lambda^a_b\, \alpha^b),$$

$$\Lambda \in SO(2,1); \qquad \text{(a)}$$

$$(\xi^a, \alpha^a) \rightarrow (\xi^a, \alpha^a + \epsilon^a_{bc}\, A^b \xi^c). \qquad \text{(b)...(4.17)}$$

The freedom of transformation (4.17a) allows us to put ξ^a into one of several distinct forms; this is then followed by the use of (4.17b) to simplify α^a. Of course, $\alpha^a \xi_a$ remains invariant. It is then seen that the pair (ξ^a, α^a) can be carried by suitable E(2,1) transformations to one of the following mutually exclusive configurations

$$\xi^a, \alpha^a \rightarrow (u, 0, 0), (\alpha, 0, 0), \; u \neq 0, \alpha \neq 0;$$

$$\rightarrow (\epsilon, 0, \epsilon), (\alpha, 0, -\alpha), \; \epsilon = \pm 1, \alpha \neq 0;$$

$$\rightarrow (0, 0, v), (0, 0, \alpha), \; v > 0, \alpha \neq 0. \qquad \text{...(4.18)}$$

Now what remains is the action by elements of SO (3,2) outside of E (2,1). Here one can convince oneself that only transformations of consequence are those generated by J_{35}, and these help to normalise α in any one of the cases (4.18) to $\epsilon' = \pm 1$:

$$[J_{35}, J_a] = 0,$$

$$[J_{35}, P_a] = P_a. \qquad \text{...(4.19)}$$

The final result is that on a rank 4 orbit in **SO** (3,2) of type *l*, there is a unique repre-

sentative element of the following mutually exclusive forms :

$$uJ_0 + \epsilon' P_0 = u\, J_{12} + \epsilon' \,(J_{03} + J_{05}), \quad u \neq 0, \epsilon' = \pm 1;$$

$$\epsilon\,(J_0 + J_2) + \epsilon'\,(P_0 - P_2) = \epsilon\,(J_{12} + J_{10}) + \epsilon'\,(J_{03} + J_{05} - J_{23} - J_{25}),$$
$$\epsilon = \pm 1, \epsilon' = \pm 1;$$

$$vJ_2 + \epsilon'\, P_2 = vJ_{10} + \epsilon'\,(J_{23} + J_{25}), v > 0, \epsilon' = \pm 1. \quad \text{.. (4.20)}$$

The third and last case of a rank 4 orbit in SO (3,2) is when ζ^A is of type s. If on such an orbit $J\,(\xi)$ is a point where the invariant vector is e_3, then $J\,(\xi) \in$ SO (2,2) associated with the dimensions 0 1 2 5. From the table of representative elements on rank 4 orbits in SO (2,2) we know that by SO (2,2) transformations $J\,(\xi)$ can be brought to one of the following standard froms labelled as in the table :

$$A : uJ_{12} + u'J_{50}, \quad u \neq 0, u' \neq 0;$$

$$B : vJ_{10} + v'J_{52}, \quad v \geqslant |\, v' \,| > 0;$$

$$C, \epsilon' : u\,(J_{12} + \epsilon' J_{50}) + v\,(J_{10} - \epsilon' J_{52}), u \neq 0, v>0, \epsilon' = \pm 1;$$

$$D, \epsilon' : u\,(J_{12} + \epsilon'\, J_{50}) + \epsilon\,(J_{10} + J_{12} - \epsilon'\, J_{50} - \epsilon' J_{52}),$$
$$u \neq 0, \epsilon = \pm 1, \epsilon' = \pm 1;$$

$$E, \epsilon' : v\,(J_{10} - \epsilon'\, J_{52}) + \epsilon\,(J_{10} + J_{12} + \epsilon'\, J_{50} + \epsilon'\, J_{52}),$$
$$v > 0, \epsilon = \pm 1, \epsilon' = \pm 1. \quad \text{...(4.21)}$$

Now these various possibilities, inequivalent within SO (2,2), can to some extent be related to one another by suitable SO (3,2) transformations. Thus a rotation of amount π in the 2–3 plane carries: u, u' under A to $-u, u'$; v, v'; under B to $v_1 - v'$: u, v, ϵ' under C to $-u, v, -\epsilon', -\epsilon'$; and a rotation of amount π in the 1-3 plane carries ϵ', u, ϵ under D to $-\epsilon', -u, -\epsilon$. In these four cases, then, we can restrict the ranges of the parameters when finding unique representative points on the concerned SO (3,2) orbits : under A, $u > 0$ and $u' \neq 0$; under B, $v \geqslant v' > 0$; under C, $\epsilon' = +1$ and $u \neq 0$, $v > 0$; under D, $\epsilon' = +1$ and $u \neq 0$, $\epsilon = \pm 1$. The type E in (4.21) does not, however admit such a reduction or havling of distinct possibilities. All we may do to reduce the number of labels is to restrict ϵ' to the value $+1$ but allow v to be nonzero positive or negative. This can be seen by tracing the effect of a rotation of amount π in the 1—3 plane on the generator in the last line of (4.21).

No simplifications beyond those described above are possible by considering elements in SO (3,2) outside SO (2,2).

Collecting all the results pertaining to the three categories t, l, s of rank 4 orbits in SO (3,2) the complete listing of possibilities can be drawn up as on p 121.

The only point of notation here requiring explanation pertains to the second, third and fourth entries. While the first letter l within $\vartheta_{3,2}$ (...) refers to the nature of ζ^A, the second letter t, l or s refers to the nature of the 3-vector ξ^a in the pair (ξ^a, α^a) describing an element in the subalgebra E (2,1).

Rank 4 orbit structure in **SO (3,2)** :

Orbit	Parameter ranges	Invariant $\mathscr{C}_1(\xi)$	Invariant $\mathscr{C}_2(\xi)$	Representative point	Invariant vector
$\vartheta_{3,2}(t; u, v)$	$u, v > 0$	$u^2 - v^2$	$-u^2v^2$	$uJ_{12} + vJ_{03}$	e_5
$\vartheta_{3,2}(l; t, u, \epsilon')$	$u \neq 0, \epsilon' = \pm 1$	u^2	0	$uJ_{12} + \epsilon'(J_{03} + J_{05})$	$e_3 + e_5$
$\vartheta_{3,2}(l; l, \epsilon, \epsilon')$	$\epsilon, \epsilon' = \pm 1$	0	0	$\epsilon(J_{10} + J_{12}) + \epsilon'(J_{03} + J_{05} - J_{23} - J_{25})$	$e_3 + e_5$
$\vartheta_{3,2}(l; s, v, \epsilon')$	$v > 0, \epsilon' = \pm 1$	$-v^2$	0	$vJ_{10} + \epsilon'(J_{23} + J_{25})$	$e_3 + e_5$
$\vartheta_{3,2}(s; A, u, u')$	$u > 0, u' \neq 0$	$u^2 + u'^2$	$u^2 + u'^2$	$uJ_{12} + u'J_{50}$	e_3
$\vartheta_{3,2}(s; B, v, v')$	$v \geq v' > 0$	$-v^2 - v'^2$	$v^2v'^2$	$vJ_{10} + v'J_{52}$	e_3
$\vartheta_{3,2}(s; C, u, v)$	$u \neq 0, v > 0$	$2(u^2 - v^2)$	$(u^2 + v^2)^2$	$u(J_{12} + J_{50}) + v(J_{10} - J_{52})$	e_3
$\vartheta_{3,2}(s; D, u, \epsilon)$	$u \neq 0, \epsilon = \pm 1$	$2u^2$	u^4	$u(J_{12} + J_{50}) + \epsilon(J_{10} + J_{12} - J_{50} - J_{52})$	e_3
$\vartheta_{3,2}(s; E, v, \epsilon)$	$v \neq 0, \epsilon = \pm 1$	$-2v^2$	v^4	$v(J_{10} - J_{52}) + \epsilon(J_{10} + J_{12} + J_{50} + J_{52})$	e_3

5. Concluding Remarks

In the preceding Sections we have exhausitively classified all the orbits under the adjoint action in each of the Lie algebras SO (p, q) for $p + q \leqslant 5$. Particular care has been taken, in view of the complexity of some of the results, to develop a suggestive and systematric notation. For each orbit, we have calculated the values of the algebraic invariants, displayed a representative element, and described the geometric nature of the latter by listing a complete set of independent vectors invariant under it.

By definition, every orbit in any of the Lie algebras admits a transitive action by the corresponding group G. Therefore it is a realisation of a certain coset space G/H, where H is the subgroups of G leaving invariant the representative point on the orbit. In the case of SO (3), as is well known this can be expressed by

$$\vartheta_3(u) \simeq \mathrm{SO}(3)/\mathrm{SO}(2). \qquad \ldots(5.1)$$

In the SO (2,1) each orbit $\vartheta_{2,1}$ $(t; u)$ is a model for SO (2,1)/SO (2); each $\vartheta_{2,1}$ $(s; v)$ realises SO (2,1)/SO (1,1); and the two orbits $\vartheta_{2,1}$ $(l; \epsilon)$ are realisations of SO (2,1)/H where H is a "parabolic" subgroup generated by $J_{12} + J_{10}$. The situation with **SO (3,1)** is actually simpler than with **SO (2,1)**. Here, each of the orbits $\vartheta_{3,1}$ $(ts; u)$, $\vartheta_{3,1}$ $(ss; v)$, $\vartheta_{3,1}$ (u, v) is a realisation of one and the same coset space SO (3,1)/SO (2) $\times$ SO (1,1) where SO (2) is generated by J_{12} and SO (1,1) by J_{03}. The single and somewhat exceptional orbit $\vartheta_{3,1}$ (ls) is the coset space SO (3,1)/N where is a two-parameter abelian group generated by $J_3 - K_1$ and $J_1 + K_3$. For the orbits in the other Lie algebras a similar through sometimes tedious analysis can be carried out by finding the stability group of the representative element in each case.

Wtih the general representation $\vartheta \simeq G/H$, the dimension of an orbit ϑ is that of G minus that of H. It is a general result that dim. ϑ is always even. For both **SO (3)** and **SO (2,1)** it is geometrically clear that each nontrivial orbit is two-dimensional. For **SO (3,1)** all orbits are of dimension four, but the situation is more complicated for both **SO (4)** and **SO (2,2)**, and also for **SO (5)**. The rank 2 orbits ϑ_5 (u) in **SO (5)** are six dimensional, since the stability group of the representative element uJ_{12} is easily seen to be the four-parameter group SO(2) $\times$ SO (3) generated by J_{12}, J_{34}, J_{45}, J_{53}. The "generic" rank 4 orbits ϑ_5 $(u, u') \subset$ **SO** (5) for $u > u' > 0$ are eight dimensional (H generated by J_{12} and J_{34}), but if $u = u'$ the dimension drops to six (H now generated by $J_{12} + J_{43}, J_{12} - J_{43}, J_{23} - J_{41}, J_{31} - J_{42}$). For **SO (4)** as well as for **SO(2,2)**, "most" orbits are four-dimenssonal, but there are some two-dimensional ones as well, in case in the cartesian product representation of an orbit one factor is trivial. Thus for example in the case of SO (4) family of orbits ϑ_4 (u, u'), the orbit dimension is a discontinuous function of the parameters u, u', since dim ϑ_4 $(u, u') = 4$ if $u > | u' |$ and dim. ϑ_4 $(u, u') = 2$ if $u = | u' |$. A similar situation occurs in **SO (5)** too.

We must note another source of discontinuity in some families of orbits as we have listed them. Thus while the presence of labelling parameters $\epsilon, \epsilon' = \pm 1$ automati-

cally means that the family of orbits concerned consists of several disjoint pieces, if a parameter u or v is only restricted to be nonzero that too results in there being several disjoint components in the family. It would have made our tables inordinately lengthy if we had insisted that each listed family of orbits form a connected set.

The availability of the vector ζ^A in the five-dimensional groups SO (5), SO (4,1) and SO (3,2) is fortunate since it immediately determines whether rank $J(\xi)$ is 2 or 4. In fact the squares of the various components ζ^A are simply the determinants of the various principal 4×4 submatrices within the 5×5 matrix (ξ_{AB}), so a vanishing (non-vanishing) ζ^A must imply rank (ξ_{AB}) is 2 (respectively 4). Further for these five-dimensional groups any orbit in the Lie algebra can be studied in the context of some suitable subalgebra since there is always at least one invariant vector. This kind of simplification does not always occur with **SO (4)** and **SO(3,1)**.

We conclude by remarking that the complete set of results obtained for the largest group we have analysed, and indeed the most intricate one, namely SO (3,2), has been used to the fullest extent in a study of a special class of optical fields[10]. We refer here to the action of general first order optical systems on the so-called Gaussian Schell-model beams, in which context the two-fold covering group Sp (4,R) of SO (3,2) plays a primary role. We refer the reader to the appropriate reference for details[10]. It is quite likely that the treatment of squeezed coherent states[1-15,43] two-photon coherent state[16-18], and generally the discussion of processes involving two modes of the photon field will be clarified as a result of our analysis.

The interested reader will find much relevant material on orbits in Sitaram and Tripathy[44], Auslander and Kostant[45] and Kirillor[46].

References

1. E. C. G. Sudarshan, and N. Mukunda, *Classical Dynamics : A Modern Perspective*, Wiley, New York, 1974.
2. R. Gilmore, *Lie Groups, Lie Algebras, and Some of their Applicatians*, Wiley, New York, 1974.
3. H. Bacry, and M. Cadilhac, *Phys. Rev.* A23 (1981), 2533.
4. M. Nazarathy, and J. Shamir, *J. Opt. Soc. Am.* 72 (1982), 356.
5. E. C. G. Sudarshan, R. Simon, and N. Mukunda, *Phys. Rev.* A28 (1983), 2921.
6. N. Mukunda, R. Simon, and E. C. G. Sudarshan, *Phys. Rev.* A28 (1983), 2933.
7. E. C. G. Sudarshan, N. Mukunda, and R. Simon, *Opt. Acta* 32 (1985).
8. N. Mukunda, R. Simon, and E. C. G. Sudarshan, *J. Opt. Soc. Am.* A2 (1985), 416.
9. R. Simon, E. C. G. Sudarshan, and N. Mukunda *Phys. Rev.* A29 (1984), 3273.
10. R. Simon, E. C. G. Sundarshan, and N. Mukunda, *Phys. Rev.* A32 (1985), 2419.
11. D. F. Walls, *Nature* 306 (1983), 141.
12. D. Stoler, *Phys. Rev.* D1 (1970), 3217.
13. J. N. Hollenhorst, *Phys. Rev.* D19 (1979) 1669.
14. G. J. Milburn, *J. Phys.* A17 (1984), 737.
15. R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, *Phys. Rev. Lett.* 55 (4985), 2409.
16. H. P. Yuen, *Phys. Rev.* A13 (1976), 2226.

17. H. P. Yuen, and J. H. Shapiro, *IEEE Trans. Inform. Theory* **IT-24** (1978), 657.
18. J. H. Shapiro, H. P. Yuen, and J. A. Machado Mata, *IEEE Trans. Inform. Theory* **IT-25** (1979), 179.
19. E. P. Wigner, *Phys. Rev.* **77** (1950), 711.
20. H. S. Green, *Phys. Rev.* **90** (1953), 270.
21. T. F. Jordan, N. Mukunda and S. V. Pepper, *J. Math. Phys.* **4** (1963), 1089.
22. N. Mukunda, E. C. G. Sudarshan, J. K. Sharma, and C. L. Mehta, *J. Math. Phys.* **21** (1980), 2386.
23. J. K. Sharma, C. L. Mehta, N. Mukunda, and E.C.G. Sudarshan, *J. Math. Phys.* **22** (1981), 78.
24. S. N. Biswas, and T. S. Santhanam, *J. Austral. Math. Soc.* **22** (1980), 210.
25. D. A. Gray, and C. A. Hurst, *J. Math. Phys.* **16** (1975), 326.
26. J. K. Sharma, C. L. Mehta, and E. C. G. Sudarshan, *J. Math. Phys.* **19** (1978), 2089.
27. T. S. Santhanam, and M. Venkata Satyanarayana, *Phys. Rev. D* **30** (1984), 2251.
28. G. M. Saxena, and C. L. Mehta, *J. Math. Phys.* **27** (1986), 309.
29. M. V. Atre, and N. Mukunda, "Classical Particles with Internal Structure: General Formalism and Application to First Order Internal Spaces", *J. Math. Phys.* **27** (1986), 2908.
30. P. A. M. Dirac, *Proc. R. Soc.* **A322** (1971), 435; **A328** (1972), 1.
31. E. C. G. Sudarshan, N. Mukunda, and C. C. Chiang, *Phys. Rev, D* **25** (1982), 3237.
32. P. A. M. Dirac, *Proc. R. Soc.* **A117** (1928), 610; **A118** (1928) 341.
33. E. Majorana, *Nuovo Cimento* **9** (1932), 335.
34. E. C. G. Sundarshan, and N. Mukunda. *Phys. Rev. D* **1** (1970), 576.
35. N. Mukunda. *Phys. Rev.* **183** (1969), 1486.
36. H. J. Bhabha, *Revs. Mod. Phys.* **17** (1945), 200.
37. H. Goldstein, *Classical Mechanics* (2nd Ed.) Addison-Wesley, Reading, MA., 1980, p. 158.
38. Throughout the paper, in the interest of brevity, the rank of the matrix $(\xi\cdots)$ will often be called the rank of the generator $J(\xi)$. Null vectors of $(\xi\cdot\cdot)$ will also often be called null vectors of $J(\xi)$. Since such vectors are invariant under the one-parameter subgroup generated by $J(\xi)$, they will also be characterised as being "invariant under $J(\xi)$".
39. In parametrising representative elements, u and u' always accompany elliptic SO (2) generrotors, v and v' always multiply hyperbolic SO(1,1) generators, and the sign parameters $\epsilon, \epsilon' = \pm 1$ multiply parabolic generators. These conventions are adopted for all the Lie algebras studied. The u terms (v terms) always contribute positively (negatively) to $\mathscr{C}_1(\xi)$, while the ϵ terms contribnte zero. The possibility of scaling the the coefficient of a parabolic generator to $\epsilon = \pm 1$ is a general feature.
40. It should be emphasized that the SO (3) transformation used here is not an element of any SO (3) subgroup of the SO (5) acting on the indices $A, B, \cdots$ Thus a similar argument is available even when one is dealing with rank 2 generators of SO (4,1) or SO (3,2), except that one loses the positive-definiteness of M in these cases.
41. Of course in the 2+2 space time like (space like) vectors have negative (positive squared norm, through the number of dimensions of each type is two.
42. The two orbits $\vartheta_{2,2}$ (ll; ϵ) in the following table are mapped onto the two orbits $\vartheta'_{2,2}$ (ll; ϵ) by the outer automorphism $M_a \leftrightarrow N_a$ of **SO (2,2)**. This operation is used later on in simplifying the catalogue of rank 4 orbits in **SO (2,2)**.
43. The role played by the pseudo-orthogonal group in squeezing becomes transparent when the problem is analysed using the Wigner distribution – R. Simon in *Symmetries in Science II* (Ed. B. Gruber), Plenum, New York, 1987.
44. B. R. Sitaram, and K. C, Tripaty, *J. Math. Phys.* **23** (1982), 206, 481, 484.
45. L. Auslander and B. Kostant, *Inv. Math.* **14** (1971), 255.
46. A. A. Kirillov, *Elements of :he Theory of Representations. Springer-Verlag Berlin* 1976.

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2. H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, Berlin, (1973) p. 283.

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